OPTIMAL FILTERING

LECTURE 5

1. Time Invariance of the Kalman Filter

2. Frequency Domain Expressions

Reading instructions: Kailath, Sect. 1.5, 14.1-14.3, 8.1-8.5, App. D.1, App. E.1-E.6 (Regarding Extended Kalman Filtering, please read 9.7, as an introduction to what is coming.)

ASSUMPTIONS TODAY

- The system described by the state space equations is **time invariant**, i.e. $F_k = F$, $G_k = G$, $H_k = H$.
- The random processes associated with the system are **stationary**, i.e.
  a) $Q_k = Q$, $S_k = S$, $R_k = R$
  b) the system is stable, $|\lambda_i\{F\}| < 1$
  c) $\Pi_0 = \bar{P}$, i.e. $\mathrm{E}\{x_0 x_0^*\} = \mathrm{E}\{\tilde{x}_{k|k-1}\tilde{x}_{k|k-1}^*\} = \bar{P}$.

This implies that the Kalman filter is time invariant!

If cItem.5) is not met, the Kalman filter will, in general, be **time varying** but will converge to a **time invariant** filter.

The process $\{x_k\}$ will be **asymptotically stationary**.

Time Invariant System

$$x_{k+1} = Fx_k + Gw_k$$
$$y_k = Hx_k + v_k$$

$\{x_k\}$ stationary $\Longrightarrow$

$$\Pi_{k+1} = \mathrm{E}\{x_{k+1}x_{k+1}^*\} = F\Pi_k F^* + GQG^*$$
$$\Pi_k = \Pi_{k+1} = \bar{\Pi}$$

Lyapunov equation: $\bar{\Pi} = F\bar{\Pi}F^* + GQG^*$

Theorem: If $[F, GQ^{*/2}]$ is controllable, $|\lambda_i\{F\}| < 1$ implies that $\bar{\Pi} = F\bar{\Pi}F^* + GQG^*$ has a unique positive definite solution. The converse is also true.

Some Definitions

Controllable (completely reachable):

$$\mathrm{rank}\begin{bmatrix} G & FG & \cdots & F^{n-1}G] \end{bmatrix} = n$$

Positive definite (PD): Let A be $n \times n$ Hermitian ($A = A^*$).

A is PD iff $x^*Ax > 0, \forall x \in \mathbb{C}^{n\times 1}, x \neq 0.$

This is equivalent to $\lambda_i(A) > 0, \forall i = 1, \ldots, n.$

Positive semidefinite (PSD): $x^*Ax \geq 0, \forall x$ or equivalently, $\lambda_i(A) \geq 0.$

A covariance matrix must be PSD. Show this!

The Lyapunov equation may be solved in a number of ways:

- linear set of equations
- infinite series
- contractive mappings

Time Invariant Kalman Filter

If the system is time-invariant and stable and the noise process is stationary, the Kalman filter converges to a time invariant filter.

$$\Longrightarrow \lim_{k\to\infty} P_{k|k-1} = \bar{P}$$

where $\bar{P}$ satisfies the "discrete-time algebraic Riccati equation" (DARE)

$$\begin{aligned}\bar{P} &= F\bar{P}F^* + GQG^* - KR_eK^* \\ &= F\left(\bar{P} - \bar{P}H^*(H\bar{P}H^* + R)^{-1}H\bar{P}\right)F^* + GQG^*\end{aligned}$$

If the filter is initialized with $\Pi_0 = \bar{P}$, it is time invariant.

Frequency Domain Expressions

Recall that for a stationary process (zero-mean) $\{a_k\}$

$$\mathrm{E}\{a_k a_l^*\} = r_a(k-l)$$

Power spectrum: $\Phi_a(z) = \sum_{k=-\infty}^{\infty} r_a(k) z^{-k}$, $\rho < |z| < \rho^{-1}$

Power spectrum of filtered signal: Transfer function $H(z)$ from $\{a_k\}$ to $\{b_k\}$

$$\Longrightarrow \Phi_b(z) = H(z)\Phi_a(z)H^*(z^{-*})$$

Power Spectrum of $\{x_k\}$ and $\{y_k\}$

Autocorrelation for $\{x_k\}$:

$$\mathrm{E}\{x_{k+l}x_k^*\} = \begin{cases} F^l\bar{\Pi} & l \geq 0 \\ \bar{\Pi}F^{*-l} & l < 0 \end{cases}$$

Power spectrum of $\{x_k\}$: $x_{k+1} = Fx_k + Gw_k$ gives transfer function $(zI - F)^{-1}G$ from $\{w_k\}$ to $\{x_k\}$. Therefore,

$$\Phi_x(z) = (zI - F)^{-1}GQG^*(z^{-1}I - F^*)^{-1}$$

Power spectrum of $\{y_k\}$: For general S,

$$\begin{aligned}\Phi_y(z) &= H(zI - F)^{-1}GQG^*(z^{-1}I - F^*)^{-1}H^* \\ &\quad + H(zI - F)^{-1}GS + S^*G^*(z^{-1}I - F^*)^{-1}H^* + R\end{aligned}$$

Special Case, $S = 0$

If $S = 0$, then

$$\begin{aligned}\Phi_y(z) &= H\Phi_x(z)H^* + R = H(zI - F)^{-1}GQG^*(z^{-1}I - F^*)^{-1}H^* + R \\ &= (I + H(zI - F)^{-1}K)(H\bar{P}H^* + R)(I + K^*(z^{-1}I - F^*)^{-1}H^*)\end{aligned}$$

Proof 1: The innovations model ($e_k = \tilde{y}_{k|k-1} = y_k - \hat{y}_{k|k-1}$)

$$\begin{aligned}\hat{x}_{k+1|k} &= F\hat{x}_{k|k-1} + Ke_k \\ y_k &= H\hat{x}_{k|k-1} + e_k\end{aligned}$$

gives transfer function $(I + H(zI - F)^{-1}K)$ from $\{e_k\}$ to $\{y_k\}$. Also, $\{e_k\}$ is temporally white with covariance $R_e = (H\bar{P}H^* + R)$.

Spectral factorization of $\Phi_y(z)$:

$$\Phi_y^+(z) = (I + H(zI - F)^{-1}K)(H\bar{P}H^* + R)^{1/2}$$

If the Kalman filter is asymptotically stable, this is minimum phase and stable.