Optimal Filtering, 2012–10–15
Due, 2010–10–23
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Homework # 7

1. Let $y(t) = x(t) + v(t)$, $t \geq 0$ where $v(t)$ is a white noise process with

$$\mathrm{E}\{v(t)v^*(s)\} = \boldsymbol{I}\delta(t-s)$$

and let

$$\dot{x}(t) = \alpha x(t) \ , \quad t \geq 0$$

where $x(0)$ is Gaussian with $\mathrm{E}\{x(0)\} = 0$ and $\mathrm{E}\{x(0)x^*(0)\} = \boldsymbol{P}_0$.

   a) Show that (for the scalar case)

$$P(t|t) = \mathrm{E}\{(x(t) - \hat{x}(t|t))^2\} = \frac{e^{2\alpha t}P_0}{1 + \frac{P_0}{2\alpha}(e^{2\alpha t} - 1)} \ .$$

   What happens as $t \to \infty$ for $\alpha > 0$ and $\alpha < 0$?

   b) Compute

$$P(s|t) = \mathrm{E}\{(x(s) - \hat{x}(s|t))^2\}$$

   for $s < t$. What happens as $t \to \infty$ for $\alpha > 0$ and $\alpha < 0$?

   c) Compare the results of parts a) and b).

2. We have

$$\begin{aligned} \dot{x}(t) &= F(t)x(t) + G(t)w(t) \qquad x(0) = x_0 \\ y(t) &= H(t)x(t) + v(t) \end{aligned}$$

$$\begin{aligned} \mathrm{E}\{x_0x_0^*\} &= \boldsymbol{\Pi}_0, & \mathrm{E}\{w(t)w^*(s)\} &= \boldsymbol{Q}(t)\delta(t-s), \\ \mathrm{E}\{v(t)v^*(s)\} &= \boldsymbol{I}\delta(t-s), & \mathrm{E}\{w(t)v^*(s)\} &= \boldsymbol{S}(t)\delta(t-s) \end{aligned}$$

$x(t)$ obeys (see e.g. Kailath "Linear Systems")

$$x(t) = \Phi(t,s)x(s) + \int_s^t \Phi(t,\tau)G(\tau)w(\tau)d\tau$$

where $\Phi(t,s)$ is the state transition matrix associated with the state space equation above. Show that

$$\mathrm{E}\{x(t)x^*(s)\} = \Phi(t,s)\boldsymbol{\Pi}(s) \ , \qquad t \geq s$$

where

$$\boldsymbol{\Pi}(t) = \mathrm{E}\{x(t)x^*(t)\} \ .$$

Note that $\Phi(t,t) = \boldsymbol{I}$.

3. Consider the same time continuous state space model as above. Show that

$$\dot{\mathbf{\Pi}}(t) = F(t)\mathbf{\Pi}(t) + \mathbf{\Pi}(t)F^*(t) + G(t)Q(t)G^*(t) \ .$$

Care must be taken of the white noise process. Follow the approach of the lecture or of the book (clearly define which notation you use!).

4. Same continuous time state space model as above. Show that

a)

$$P(t,s) = \mathrm{E}\{\tilde{x}(t|t)\tilde{x}^*(s|s)\} = P(t,t)\Phi_k^*(s,t) \quad s \geq t$$

where $\Phi_k(t,s)$ is the state transition matrix associated with

$$\dot{\tilde{x}}(t) = (F(t) - K(t)H(t))\tilde{x}(t) + G(t)w(t) - K(t)v(t)$$

i.e.

$$\tilde{x}(t) = \Phi_k(t,s)\tilde{x}(s) + \int_s^t (\Phi_k(t,\tau)G(\tau)w(\tau) - \Phi_k(t,\tau)K(\tau)v(\tau))d\tau, \quad t \geq s$$

b)

$$\hat{x}(t|t_f) = \hat{x}(t|t) + P(t|t)\lambda(t)$$

where

$$\begin{aligned} \lambda(t) &= \int_t^{t_f} \Phi_k^*(s,t)H^*(s)R_e^{-1}(s)e(s)ds \\ \dot{\lambda}(t) &= -\left(F(t) - K(t)H(t)\right)^* \lambda(t) - H^*(t)R_e^{-1}(t)e(t) \\ \lambda(t_f) &= 0 \end{aligned}$$

c) When $S(t) \equiv 0$

$$\dot{\hat{x}}(t|t_f) = F(t)\hat{x}(t|t_f) + G(t)Q(t)G^*(t)P^{-1}(t|t)\left(\hat{x}(t|t_f) - \hat{x}(t|t)\right), \quad t \leq t_f$$

You have just derived some smoothing formulas!