OPTIMAL FILTERING

LECTURE 9

- Implementational and computational aspects
  - Information form of the Kalman filter
  - Square root algorithms
  - Fast algorithms

Reading instructions: Kailath, Sect. 9.5, 12.1-12.4, 11.1

INFORMATION FILTER

- Initial state completely unknown $\iff P_0 = \infty$. Hard to handle!
- Trick: $P_0^{-1} = 0$ can be handled, if we propagate $P_k^{-1} \iff$ the **information** formulation of the Kalman filter.
- Tool: **Matrix Inversion Lemma**

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1}DA^{-1}$$

Kalman in Information Form

Assuming $S_k = 0$ (extension not difficult)

$$\begin{aligned}
P_{k|k}^{-1} &= P_{k|k-1}^{-1} + H_k^* R_k^{-1} H_k \\
P_{k+1|k}^{-1} &= (I - B_k G_k^*) A_k \\
\hat{a}_{k+1|k} &= (I - B_k G_k^*) F_k^{-*} \hat{a}_{k|k} \\
\hat{a}_{k|k} &= \hat{a}_{k|k-1} + H_k R_k^{-1} y_k \\
A_k &= F_k^{-*} P_{k|k}^{-1} F_k^{-1} \\
B_k &= A_k G_k (G_k^* A_k G_k + Q_k^{-1})^{-1}
\end{aligned}$$

with transformed state variables:

$$\begin{aligned}
\hat{a}_{k|k-1} &= P_{k|k-1}^{-1} \hat{x}_{k|k-1} \\
\hat{a}_{k|k} &= P_{k|k}^{-1} \hat{x}_{k|k}
\end{aligned}$$

Compare complexity: $(G_k^* A_k G_k + Q_k^{-1})^{-1} \Longleftrightarrow (H_k P_{k|k-1} H_k^* + R_k)^{-1}$

Square Root Algorithms

- P_k may become indefinite because of numerical errors, may cause error propagation.
- Factor $P_k = P_k^{1/2} P_k^{*/2}$ and find recursions for $P_k^{1/2}$.
- Implicitly guarantees that P_k is positive semidefinite.
- Condition number of $P_k^{1/2}$ is square root of condition number of P_k.

Square Root Algorithm

Time update: Find orthogonal matrix Θ, ($\Theta\Theta^* = I$) so that

$$\begin{bmatrix} F_k P_{k|k}^{1/2} & G_k Q_k^{1/2} \end{bmatrix} \Theta = \begin{bmatrix} X & 0 \end{bmatrix}$$

Then $P_{k+1|k}^{1/2} = X$!

Measurement update: Find orthogonal matrix Θ, ($\Theta\Theta^* = I$) so that

$$\begin{bmatrix} R_k^{1/2} & H_k P_{k|k-1}^{1/2} \\ 0 & P_{k|k-1}^{1/2} \end{bmatrix} \Theta = \begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix}$$

then

$$R_{e_k}^{1/2} = X, \qquad P_{k|k}^{1/2} = Z, \qquad \tilde{K}_k = K_k R_{e_k}^{1/2} = Y$$

Fast Algorithms

Assumption: All system matrices time invariant

Chandrasekhar Equations: Let $\delta P_k = P_{k+1|k} - P_{k|k-1}$, $k \geq 0$, then

$$\delta P_{k+1} = (F - K_{k+1}H)(\delta P_k + \delta P_k H^* (H P_{k|k-1} H^* + R)^{-1} H \delta P_k)(F - K_{k+1}H)^*$$

Rank result: $\text{rank}[\delta P_{k+1}] \leq \text{rank}[\delta P_k] \leq \cdots \leq \text{rank}[\delta P_0]$

How to exploit: $\delta P_0 = L_0 M_0 L_0^*$, $\text{rank}[M_0] = \text{rank}[\delta P_0] = \alpha$.

Update L_k, M_k, see Kailath, Section 11.1.

Computational complexity: $\mathcal{O}(\alpha n^2)$ instead of $\mathcal{O}(n^3)$