1. Repetition – probability theory and transforms

1.1. A prisoner is kept in a cell with three doors. Through one of them he can get out of the prison. The other one leads to a tunnel: through this he is back to the cell after one day of walking. The third door leads to another tunnel, through which it takes the prisoner three days to get back to the cell. Assume that the prisoner selects a door randomly every time he gets back to his cell. Calculate the expected time it takes for the prisoner to get out of the prison.

1.2. X is a random variable chosen from X_1 with probability *a* and from X_2 with probability *b*. Calculate E[X] and σ_X for a=0.2, b=0.8, X_1 is an exponentially distributed r.v. with parameter $\lambda_1=0.1$ and X_2 is an exponentially distributed r.v. with parameter $\lambda_2=0.02$. Let the r.v. Y be chosen from D_1 with probability *a* and from D_2 with probability *b*, where D_1 and D_2 are deterministic r.v.s. Calculate the values D_1 and D_2 so that E[Y]=E[X] and $\sigma_Y=\sigma_X$.

1.3. X is a discrete stochastic variable, $p_k = P(X = k) = \frac{a^k}{k!}e^{-a}$, (k=0,1,2…), and *a* is a positive constant.

a) Prove that $\sum_{k=0}^{\infty} p_k = 1$.

b) Determine the z-transform (generating function) $P(z) = \sum_{k=0}^{\infty} z^k p_k$.

c) Calculate E(X), Var(X), and E{X(X-1)…(X-r+1)}, r=1,2…., with and without using z-transforms.

1.4. X_i's are Poisson distributed random variables, thus $p_k = P(X_i = k) = \frac{a_i^k}{k!}e^{-a_i}$, (k=0,1,2…) and a_i's are a positive constant (i=1,2,…n). X_1, X_2, … X_n are assumed to be independent. Give the probability distribution function of $X = \sum_{i=1}^{n} X_i$.

1.5. X is a positive continuous stochastic variable with probability distribution function (PDF)

$F(x) = P(X \le x) = \begin{cases} 0 & (x < 0) \\ 1 - e^{-ax} & (x \ge 0) \end{cases}$, where *a* is a positive constant.

a) Give the probability density function $f(x) = \frac{dF(x)}{dx}$.

b) Give $\overline{F}(x) = P(X > x)$.

c) Calculate $F^*(s) = E(e^{-sx}) = \int_0^{\infty} e^{-sx} f(x)dx$.

d) Calculate the expected values m=E(X), $E(X^k)$ (k=0,1,…),
the variance $\sigma^2 = Var(X) = E\{(X-m)^2\}$,
the standard deviation $\sigma = \sqrt{Var(X)}$ and
the coefficient of variation $c = \frac{\sigma}{m}$, with and without the transform F*(s).

1.6. X_i's are independent, exponentially distributed random variables with a mean value of 1/a (a>0) (i=1,2,…n). Calculate $P(X \le x)$ and $P(X \ge x)$, where

a) $X=\min(X_1, X_2,…X_n)$

b) $X=\max(X_1, X_2, \ldots X_n)$.

1.7. X_i's are independent, exponentially distributed random variables with a mean value of 1/a (a>0) (i=1,2,…n).

a) Determine the distribution of $X = \sum_{i=1}^{r} X_i$.

b) Give f(x), the probability density function (pdf) of X;
F(X), the probability distribution function (PDF) of X;
the function $\overline{F}(x) = P(X > x)$ and
the Laplace transform of X.

c) Calculate the mean value m=E(X),
the variance $\sigma^2 = Var(X)$,
the standard deviation σ and
the coefficient of variation $c = \sigma/m$.

1.8. Prove the memoryless property of the exponential distribution

2. Poisson process

2.1. Consider a Poisson arrival process. Show that the interarrival times are exponential.

2.2. A multiplexer has *k* incoming and one outgoing links. Packets arrive on each incoming link according to a Poisson process with intensities $\lambda_{1}, \ldots \lambda_{k}$, the processes are independent. Prove that the combined packet stream is still Poissonian with $\lambda = \sum_{i=1}^{k} \lambda_i$.

2.3. A router implements traffic dispersion, i.e., it spreads incoming packets over *r* outgoing links. Packets arrive according to a Poisson process with rate λ, and outgoing links are selected randomly with probability α_{i}. Prove that the packet streams on links 1…*r* are also Poissonian. Give the packet transmission rate on each outgoing link.

2.4. Packets arrive to a node according to a Poisson process with an average rate of 1000 packets per second.

a) What is the probability that no packet arrives in a one-millisecond period?

b) Let *t* be an arbitrary point in time and *T* the elapsed time until the 500th packet arrives after *t*. Find the expected value and variance of *T*.

2.5. Let $\{X_j\}$ be a sequence of independent "Bernoulli" random variables, i.e., $X_j \in \{0,1\}$. Let $\Pr(X_j = 1) = p$ and consequently $\Pr(X_j = 0) = 1 - p$. Let $S_N = X_1 + ... + X_N$ be the sum of the N consecutive variables, where N has a Poisson distribution with a mean λ. Show that S_N has a Poisson distribution with mean λp.

2.6. Pure Aloha is a medium access control protocol originally developed at the University of Hawaii for packet radio networks. In this protocol, when a node has a packet to send, it transmits it immediately. If the transmitted packet collides with other packets, all packets are lost. The nodes packets will be retransmitted after random delays. Calculate the throughput of the pure Aloha system.

2.7. Prove that Poisson arrivals see time averages (PASTA).

3. Balance equations, birth-death processes, continuous Markov-chains

3.1. Consider a pure birth process, assuming, that $\lambda_k=\lambda$ for all k=0,1,2…Calculate the probability that the system is in state k at time t, given that $P_0(0)=1$.

3.2. Consider a birth-death process with three states, where the transition rate from state 2 to state 1 is $q_{21}=\mu$ and from state 2 to state 3 is $q_{23}=\lambda$. Show that the time spent in state 2 is exponentially distributed with mean $1/(\lambda+\mu)$.

3.3. Assume that the number of call arrivals between two locations has Poisson distribution with intensity λ. Also assume that the holding times of the conversations are exponentially distributed with a mean of $1/\mu$. Calculate the average number of arriving calls for a period of a conversation.

3.4. Consider a communication link with a constant rate of 4.8kbit/s. Over the link we transmit two types of messages, both of exponentially distributed size. Messages arrive in a Poisson fashion with $\lambda=10$ messages/second. With probability 0.5 (independently from the previous arrivals) the arriving message is type 1 and has a mean length of 300 bits. Otherwise a message of type 2 arrives, with a mean length of 150 bits. The buffer at the link can at most hold one message of type 1 or two of type 2. A message being transmitted still takes up a place in the buffer.

a) Determine the mean and the coefficient of variation of the service time of a randomly chosen arriving message.
b) Determine the average times in the system for accepted messages of type 1 and 2.
c) Determine the message loss probabilities for messages of type 1 and 2.

3.5. Consider a Markovian system with discouraged job arrivals. Jobs arrive to the server in a Poisson fashion, with an intensity of one job per 7 seconds. The jobs observe the queue. They do not join the queue with probability l_k if they observe k jobs in the queue. $l_k=k/4$ if $k<4$ and 1 otherwise. The service time is exponentially distributed with mean time of 6 seconds.

a) Determine the mean number of customers in the system, and
b) the number of jobs served in 100 seconds.

3.6. Consider a network node that can serve 1 and store 2 packets altogether. Packets arrive to the node according to a Poisson process. Serving a packet involves two independent sequentially performed tasks: the error check and the packet transmission to the output link. Each task requires an exponentially distributed time with an average of 30 msec. Given, that we observe that the node is empty in 60% of the time, what is the average time spent in the node for one packet?

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3.7. Consider a birth-death process with two states and transition intensities $q_{01}=\lambda$ and $q_{10}=\mu$. Let $\lambda=2\mu$. In state 0 there are on average a_0 arrivals per time unit, in state 1 there are on average a_1 arrivals per time unit.

a) Calculate the average number of arrivals in an interval of length t.
b) Calculate the transient state probabilities for t=1 and t=10, $\mu=1$, and P(0)=[1 0].

4. Queuing systems

4.1. Packets are arriving to a communication node with a single output link according to a Poisson process. Give the Kendall notation for the following cases:

a) the packet lengths are exponentially distributed, the buffer capacity at the node is infinite;
b) the packet length is fixed, the buffer can store *n* packets;
c) the packet length is *L* with probability p_L and *l* with probability p_l; there is no buffer at the node.

4.2. Give the Kendall notation of the following systems. Telephone calls arrive to a PBX with *C* output links. The calls arrive as Poisson process, the call holding times are exponentially distributed.

a) Calls arriving when all the output links are busy are blocked.
b) Up to *c* calls can wait when all the output links are blocked.

4.3. Why it is not a good idea to have a G/G/10/12/5 system?

4.4. Which system provides the better performance, an M/M/3/300/100 or an M/M/3/100/100?

4.5. A PBX was installed to handle the voice traffic generated by 300 employees in an office. Each employee, on average, makes 2 calls per hour with an average call duration of 4.5 minutes. The PBX has 90 outgoing links.

a) What is the offered load to this PBX?
b) What is the utilization of the outgoing links? Assume that calls arriving when all the links are busy are queued up.

4.6. Jobs arrive to a service facility with 15 servers, on average 30 jobs per second. The mean service time of a job is 0.25 seconds.

a) The facility is equipped with an infinite queue, and the average number of jobs in the queue is 60. Calculate the average time spent in the system.
b) If we limit the queue capacity to 30, the loss probability will be 0.05. Calculate the average number of customers in the servers. The average time spent in the system is 4 seconds. Calculate the mean number of customers in the system.

5. M/M/1 queuing systems

5.1. In a computer network a link has a transmission rate of C bit/s. Messages arrive to this link in a Poisson fashion with rate λ messages per second. Assume that the messages have exponentially distributed length with a mean of $1/\mu$ bits and the messages are queued in a FCFS fashion if the link is busy.

a) Determine the minimum required C for given λ and μ, such that the average system time (service time + waiting time) is less than a given time T_0. Find the probability that the total time a message spends in the system is larger than $3T_0$.

b) Find the minimum required C for fixed λ and μ, such that the probability of the system time being larger than a given t is less than p ($Pr\{\text{system time} > t\} < p$).

c) Determine the maximum allowed message arrival rate λ for fixed C and μ, such that the probability of the system time being larger than a given t is less than p.

5.2. Consider an M/M/1 system.

a) Assume FCFS service at the queue. Determine the average time T an arbitrary message spends in the system, with the following reasoning: a message that arrives to the system and finds k messages in the queue, will stay in the system for k+1 service times.

b) Calculate T with a similar reasoning as before, assuming LCFS service strategy.

c) Determine T for an arbitrary queue service strategy.

d) Calculate Var(N), the variance of the number of customers in the system.

(FCFS: First Come First Served, LCFS: Last Come First Served)

5.3. Consider an M/M/1 system with parameters λ and μ where customers are impatient. As the customers arrive to the system, they estimate their waiting time ω, and leave the system with probability $1-e^{-\alpha\omega}$ ($\alpha \geq 0$). They estimate the waiting time as $\omega = k/\mu$ if they find k customers in the system.

a) Determine p_k as a function of p_0.

b) Determine p_k in explicit form and also the mean number of customers in the system for $\alpha \to \infty$.

5.4. Consider an M/M/1 system where the server becomes stressed and less effective if a queue builds up. If there is no queue, the service time is exponentially distributed with a mean of $1/\mu$. If there are waiting customers, the service time gets r times longer in average. Customers arrive to the system with intensity λ.

a) Draw the state diagram.

b) Let $P(z) = \sum_{k=0}^{\infty} z^k p_k$. Calculate $P(z)$ for the above system.

c) Calculate the mean number of customers in the system with the help of $P(z)$.

d) What is the probability that a random observer sees k customers in the system, and what is the probability that an arriving customer finds k customers in the system?

5.5. Consider a queuing system with a single server. The arrival events can be modeled with Poisson distribution, but two customers arrive to the system at each arrival event. Each customer requires an exponentially distributed service time.

a) Draw the state diagram.

b) Determine p_k (k=0,1,….) using local balance equations.

c) Let $P(z) = \sum_{k=0}^{\infty} z^k p_k$. Calculate $P(z)$ for the system. Note, that $P(z)$ must be finite for $|z| < 1$, and we know that $P(1) = 1$.

d) Calculate the mean number of customers in the system with the help of $P(z)$, and compare it with the one of the M/M/1 system.

5.6. A queuing system has one server and infinite queuing capacity. The number of customers in the system can be modeled as a birth-death process with $\lambda_k=\lambda$ (k=0,1,2,…) and $\mu_k=k\mu$ (k=0,1,2,…), thus the server increases the speed of the service with the number of customers in the queue. Calculate the average number of customers in the system as a function of $\rho=\lambda/\mu$.

5.7. Customers arrive to a single server system in groups of 1, 2, 3 or 4 customers, the number of customers per group is i.i.d.. There are in total 4 places in the system. If a group of customers does not fit into the system, none of the members of the group joins the queue. 10% of the customers arrive in a group of 1, 20% in a group of 2, 30% in a group of 3 and 40% in a group of 4 customers. The average number of arriving customers is 75 per hour, the interarrival time between groups is exponentially distributed. The service time is exponentially distributed with mean 0.5 minute.

a) Give the Kendall notation of the system and draw the state diagram.
b) Calculate the average number of customers in the queue and the mean waiting time per customer.
c) Calculate the probability that the system is full and the probability that a customer arriving in a group of *k* customers can not join the queue.
d) Calculate the probability that an arriving customer (in general) can not join the queue, and the probability that an arriving group of customers can not join the queue. Explain the results.
e) What is the average waiting time for a customer arriving in a group of 3 customers?

6. M/M/m/m loss systems

6.1. Consider an M/M/m loss system with ρ Erlang offered traffic. Servers are numbered as 1,2,…m and are selected in this order. Show that the expected part of time that the first m-1 servers are busy but the mth one is free is $E_{m-1}(\rho) - E_m(\rho)$.

6.2. Consider an M/M/3 loss system with $\lambda=1s^{-1}$ and $1/\mu=2s$.

a) Calculate the time blocking probability.
b) Calculate the mean number of blocked customers per hour.
c) Give the offered load, the effective load and blocked traffic in the system.

6.3. We consider two types of call arrivals to a cell in a mobile telephone network: new calls that originate in the cell, and calls that are handed over from neighboring cells. It is desirable to give preference to hand-over calls over new calls. For this reason, some of the channels in the cell are reserved for hand-over calls, while the rest of the channels are available to both types of calls. For the questions below assume the following. Channels in the cell are held for two minutes in average, with exponential distribution. All calls arrive according to a Poisson process with intensity λ_{nc}=125 calls per hour for new calls, and λ_{ho}=50 calls per hour for hand-over calls. The cell has 10 channels; a call occupies one channel.

a) Draw a state diagram of the channel occupancy in a cell when 2 channels are used exclusively for hand-over calls.
b) Calculate the blocking probability in the cell if no channels are reserved for hand-over calls. What is the average number of channels used?
c) Find the minimum number of channels reserved for hand-over calls so that their blocking probability is below 1 percent. What is the blocking probability for new calls in this case?

6.4. Consider two loss systems with offered load ρ_1 and ρ_2. The first system has m_1 servers and the second has m_2 servers. Compare these systems to a third one with m_3 servers and $\rho_1+\rho_2$ offered traffic. Assume that ρ_1=3 Erlang and ρ_2=6 Erlang. The interarrival times and service times are exponentially distributed.

a) Calculate the required number of servers in the three systems to achieve a time blocking probability less than 0.1%.
b) Calculate the average number of jobs in the system.
c) Give the average load per server for the three systems.
d) Calculate the probability that at most two servers are free. Do the calculation for all three systems.

6.5. A telephone switch has 10 outbound lines and a large number of incoming lines. Upon arrival a call on the input line is assigned an outbound line if such a line is available – otherwise the call is blocked and lost. The outbound line remains assigned to the call for its entire duration which is of exponentially distributed length. Assume that 180 calls/hour arrive in Poisson fashion whereas the mean call duration is 110 seconds.

a) Determine the blocking probability.
b) How many calls are rejected per hour?
c) What is the average load per server (in Erlang)?
d) What is the maximum arrival rate at which a blocking probability of 2% can be guaranteed?

6.6. (The repairman model) There are K computers in an office, and a single repairman. Each computer breaks down after an exponentially distributed time with parameter α. The repair takes an exponentially distributed amount of time with parameter β. Only one computer is being repaired at a time, computers break down independently of the repair process, and repair times and lifetimes of the computers are independent.

a) What is the probability that i computers are working at time t ?
b) What is the average failure rate (i.e. the average number of computers that fail per time unit)?
c) How much percent of the time is the repairman busy?
d) How much percent of the time are all computers broken?
e) How many computers should we have if we would like to have K computers to work on average?

7. M/M/m systems

7.1. At a Swedish liquor store, on the average 100 customers arrive every hour (in a Poisson fashion). Each customer requires service with a duration that is exponentially distributed. Three counters are planned to be opened. The Management is contemplating two different queuing disciplines: A: each customer upon arrival randomly selects a queue and stays with that queue until he is served. The mean service time in this case is 45 seconds. B: upon arrival, he receives a numbered tag and is served in a strict FCFS order. Since he has to walk to the free counter when his number comes up the mean service time will be 50 seconds. Determine the average time in the system for both strategies. Which solution should the management choose?

7.2. The performance of a system with one processor and another with two processors will be compared. Let the interarrival times of jobs be exponentially distributed with parameter λ . We consider first the system with one processor. The service time of the jobs is exponentially distributed with a mean of 0.5 sec.

a) For an average response time of 2.5 sec (the total time of a job in the system), how many jobs per second can be handled?
b) For an increase of λ with 10%, how much will the response time increase?
c) Calculate the average waiting time, the average number of customers in the server and the utilization of the server. What is the probability of the server being empty?

Let us now compare this system with a system with two cheaper processors, each with a mean service time of 1 sec.

d) How many jobs can now be handled per second with a mean response time of 2.5 sec?
e) For an increase of λ with 10%, how much will the response time increase?
f) Calculate the average waiting time, the average number of customers in the server and the utilization of the server. What is the probability of both of the servers being busy?

7.3. Customers arrive to an M/M/3 system with intensity 1.6 arrivals per time unit. The average service time is 1.25 time units. The service strategy is FCFS. Consider a customer that finds all 3 servers busy and 2 customers waiting in the queue. Calculate the probability that this customer has to wait longer than 1.25 time units.

7.4. Three different types of customers arrive to a system with *m* servers and a shared queue. The interarrival times between consecutive arrivals of a job of type *i* are exponentially distributed with a mean of $1/\lambda_i$ (i=1,2,3). The service times are exponentially distributed with a mean of $1/\mu$.

a) Draw the state diagram, where the states give the number of customers in the system, *k*, *k*=0,1,2,…
b) Express the state probabilities with μ, λ_1, λ_2, λ_3 and *k*.
c) Calculate the average number of customers in the system for m=2, λ_1=3/minute, λ_2=2/minute, λ_3=1/minute and $1/\mu$=10sec.
d) Calculate the average number of customers of type 1 in the system.
e) Calculate the average waiting time for an arbitrary customer.
f) Calculate the average waiting time for a customer of type 1.

7.5. Consider an $M/M/\infty$ system with arrival rate λ and mean service time $1/\mu$. Calculate the following system parameters:

a) served traffic,
b) mean waiting time,
c) state probabilities,
d) average number of customers in the system.

7.6. Consider a pure delay system where customers arrive according to a Poisson process with intensity λ=3. The service time is exponentially distributed with mean value 1/3. The queuing discipline is FCFS. There are two servers in the system, and one of them is always available. The other one starts service when the queue length would become two (so that it immediately becomes one). If there are no more customers in the queue, the server which becomes idle first is closed (and stays closed until the queue length becomes two again). Let us denote the state of the system with (i,j), where i is the total number of customers in the system and j is the number of open servers.

a) Give the Kendall notation of the system and draw the state transition diagram.
b) Find the steady state distribution.
c) Calculate the average queue length and the average number of customers in the servers.
d) Give the waiting time distribution of a customer arriving in state (i,2), where $i \geq 3$.
e) Give the waiting time distribution of a customer that arrives in state (1,1) of the system and thus brings it into state (2,1)

8. M/M/m with limited system capacity and limited number of customers (M/M/m/S and M/M/m//K)

8.1. We investigate the traffic on an output link of a network node. We model the transmission link as an M/M/1/K system with arrival rate λ message/second and mean service time (message transmission time) $1/\mu$ second. Calculate the probability that a message arriving to the transmission link

a) is transmitted immediately,
b) has to wait but will be transmitted,
c) is dropped.

8.2. Consider an M/M/2 system with a single queuing position (M/M/2/3). Customers that arrive when both servers are busy but the queue position is free, wait until served. Customers arriving when the servers and the queue position are all occupied leave the system. Calculate the probability that a customer has to wait and the probability that it leaves the system, given that the arrival rate is 2 arrivals per minute and the average service time is 45 seconds. Calculate the probability that a customer has to wait longer than *t*, before it is served.

8.3. We consider an M/M/m system with parameters λ and $1/\mu$. The system sets the number of active servers in the following way. Only one server is active if there are 0,1 or 2 customers in the system, and 2 servers are active otherwise.

a) Draw state transition diagram.
b) Determine the state probabilities p_k in steady state (k=0,1,2….).
c) Calculate the average number of customers under service. (Before you start to calculate, think a bit.)

8.4. Consider a Markovian queuing system with two servers and two queuing positions. There are five sources in the system. The sources generate calls with intensity β if they do not have call in progress (waiting or under service). The service rate is μ.

a) Draw the state transition diagram.
b) Calculate the mean waiting time in the queue.
c) Calculate the time blocking probability.
d) Calculate the call blocking probability.

For b), c) and d) assume that $\beta=\mu=1s^{-1}$.

8.5. A dormitory has a local area network that interconnects all residents. A group of eight students has agreed to get some shared dial-up links for their Internet access. Osqualda is one of them. She has studied queuing theory and volunteers to calculate and order the needed number of links. After a small survey she finds that the group requires that the probability of blocked calls should not exceed two percent during the busiest evening hours. a call to the Internet service provider occupies one link for an average duration of 15 minutes, and the lengths are exponentially distributed. The members of the group start their sessions independently of one another, and each one has an exponentially distributed time between call attempts with mean of 75 minutes.

a) How many links should Osqualda at least order?
b) Osqualda estimates that the full population of almost 300 students in the dormitory places an average of 800 calls per hour to the Internet providers during busy hours. How many links would suffice for the entire dormitory given that the call durations and blocking probability would remain the same? She also thinks of installing a queue for call requests to avoid blocking. What would the average waiting time be to get access to a link in the corresponding wait system (M/M/m)?

8.6. In the kitchen of a dormitory corridor there are 2 hobs for cooking and 3 places on the sofa. There are 8 students living in the corridor, each of them goes on average every $1/\alpha$ hours to the kitchen to cook (if he is not cooking already), the interarrival time is exponentially distributed. If on arrival the 2 hobs are occupied, the student looks for a place on the sofa, if the sofa is occupied as well, he goes back to his room and tries again later. Students spend an exponentially distributed amount of time cooking with mean $1/\beta$. α=0.5/hour, β=1/hour.

a) Draw the state transition diagram.
b) Calculate the mean waiting time of the students.
c) Calculate the ratio of time the kitchen is completely full, e.g. a student arriving has to go back to his room.
d) Calculate the probability that a student finds the kitchen completely full.
e) Calculate the probability that a student has to wait more than 2 hours (supposed that he can sit down in the kitchen).

9. Stage method - the Erlang and Hyperexponential distributions

9.1. Consider an $M/E_r/1$ system without buffer. Let the state j be the number of stages left of service, and P_j be the equilibrium probability that the system is in state j.

a) Find the value of P_j for j=0,1,....r.

b) Find the probability of a busy system.

9.2. Consider an $M/H_2/1$ system, without buffer. The branching probabilities are α and $1-\alpha$ to servers 1 and 2, and the service rates are $\mu_1=2\mu\alpha$, $\mu_2=2\mu(1-\alpha)$.

a) Calculate the steady state probability that the system is idle.

b) What is the probability of the first service unit being busy?

c) What is the probability that the system is busy?

9.3. As we arrive to the post office we observe that two counters are open. At the first counter there are two people (including the one being served) whereas there is only one person at the second counter. Assume that the service times are independent and exponentially distributed with mean of 1 minute.

a) What are the expected waiting times given that you choose the first or the second counter?

b) Assume that you choose the second counter. What is the probability that this was a wrong decision, i.e., you would have been served earlier at the first counter?

c) What is the probability in b) if Ms Smith is the customer at the second counter, and her service times are exponentially distributed with a mean of 2 minutes?

9.4. A small Swedish company has just begun to use e-mail as a means of communicating with its office in Moscow. Because the Internet is not available in Moscow, they resort to using a modem. The phone lines in Moscow are not that reliable and they can only get a connection at 1200bits/second. In order to keep the phone bills at a reasonable level, the e-mails are not transmitted immediately, but stored in a mail buffer. A connection is established when exactly 5 messages are ready for transmission in the buffer. With the assumption that mail messages arrive according to a Poisson process with an average rate of 3 messages per hour, find the following.

a) What is the probability that a message arriving to an empty buffer will wait in the buffer for more than 2 hours before being transmitted?

b) There are only two kinds of messages with exponentially distributed message length, a short memo with average length of 1000 bits and a long report with average length of 10000 bits. 75% of the messages are short ones. The charges to Moscow are 10:20 SEK per minute. In addition to the actual data transmission, a 7 second connect time is required to establish each connection in error correcting mode, thus no messages have to be retransmitted. What is the average cost for this mail service during a regular office day of 10 hours?

9.5. The service times in a server are Erlang-r distributed with mean $1/\mu$. What is the value a of the service time for which $P(a-\varepsilon<x<a+\varepsilon)$, $\varepsilon>0$ constant, is maximal? What happens if $r \rightarrow \infty$?

10. M/G/1 queues

10.1. Consider an M/G/1 system, with the following service strategy. As a job enters the server, its service time is generated randomly. The service time will be zero seconds with probability *p*, otherwise the service time is selected according to the exponential distribution pe^{-px}, $x \geq 0$.

a) Determine the average service time and the variance of the service time.
b) Calculate the expected waiting time and give $W^*(s)$.
c) Based on b) calculate the expected waiting time and the function $W(t) = P(\text{waiting time} \leq t)$.
d) Calculate the z-transform of the number of customers in the queue, and check the result by calculating the expected number of customers and the mean waiting time.

10.2. An M/M/1 queue has 8.1 packets waiting for service in steady state.

a) What is the average number of packets in an $M/E_5/1$ system with the same utilization? What queuing system does it tend to as the number of stages gets large?
b) How should service times μ_1 and μ_2 be chosen for the $M/H_2/1$ queue to have the average number of packets in the system be 15 at the same utilization? The average arrival rate is 1800 packet per second and μ_1 is selected with probability 0.2.

10.3. Packets at a source are processed in three steps: the payload is encrypted in the first step, then the header is generated and, finally, the packet is transmitted. Assume that the duration of each step is exponentially distributed with a mean of 30 msec. The generation of packets is Poissonian with one packet per 120 msec in average.

a) What is the probability that the node is idle?
b) What is the average queue size at the node?

10.4. Messages are transmitted with a stop and wait ARQ protocol, where error control symbols are used to detect erroneous transmissions. If a message is declared correct by the receiver it will reply by transmitting an ACK-message. If an error is detected a NACK message is sent. Receiving an ACK the transmitter removes the original message from its transmission queue and transmits the next message. If a NACK is received, the original message is retransmitted (until an ACK is received). Assume that all message transmissions require 1 time unit, and ACK/NACK transmissions are of neglectable duration. Messages arrive at the transmitter in a Poisson fashion with rate λ. Assume that a transmitted message is received with error with probability *p*, independently from other transmissions.

a) What is the probability distribution of the number of transmissions required to forward one message?
b) What is the expected message delay?
c) What is the maximum allowed arrival rate if the system has to remain stable?

10.5. Consider a telephone answering service company with one employee. Calls arrive in a Poisson fashion with a rate of 3 calls/minute. If the employee is occupied with a call, arriving calls are automatically put on hold in an FCFS queue. Assume further that for ¾ of the calls the caller just leaves his name, which takes an exponentially distributed time with mean of 10 sec. For other calls, the caller leaves longer message, which takes an exponentially distributed time with mean of 20 sec.

a) What is the mean waiting time before service for an arriving customer?
b) What is the probability that an arriving customer must wait for more than 2 minutes before service?

11. M/G/1 queues with vacation and priorities

11.1. A communication link serving two users is used in an asynchronous TDM mode. Each user generates messages of constant size which all fit into a time slot of the TDM system. The messages arrive at the link in a Poisson fashion with rate 0.4 and 0.2 messages/slot respectively. Messages that cannot be transmitted immediately are queued in two buffers one for each user of unlimited capacity. Assume that odd slots are exclusively assigned to user 1 whereas even slots may be used only by user 2.

a) What are the expected delays of messages from user 1 and 2 respectively?
b) What is the expected delay of a randomly chosen message on the link?

11.2. A communication link can be shared between 10 users in an asynchronous TDM fashion. Each user generates messages of constant size, which fits into a time slot. The messages arrive Poisson distributed in time with rate $\lambda/10$ for each user. Messages that cannot be transmitted immediately are queued in buffers. Consider two transmission strategies:

A) Each user is assigned every 10^{th} time slot, which may be used exclusively by the user.
B) Each user adds an address to his message, which now become 20% larger, thus the system slot size has to be increased as well (fewer slots per time unit!). The messages of all users are then transmitted in a strict FCFS order.

a) For which λ's is system B) better than A)?
b) What is the maximum traffic load for a stable system for the two strategies?

11.3. Two types of packets (A and B) arrive at a switching node as independent Poisson processes with mean arrival rates of 100 packets/s for type A and 20 packets/s for type B. Type A packets have a constant length of 20 bits. Type B packets are of exponential length with a mean of 100 bits. The outgoing link is operating at 10 000 bit/s.

a) Determine the waiting time and total time spent at the switch of the two types of packets for a non-preemptive priority system, if type A packets have higher priority. Repeat the calculation when the priorities are switched.
b) Determine the waiting time and system time spent at the switch of the two types of packets for a preemptive-resume priority system, if type A packets have higher priority

11.4. A network carries packets with two priorities. Packets of high and low priorities are enqueued in separate and infinite buffers in a node. The service is in priority order without preemption. High priority packets are of a fixed length, 100 bytes, and arrive as a Poisson process with an average of 1000 packets per second. The lengths of the low-priority packets are exponentially distributed with a mean of 250 bytes. The maximum service rate is 2 Mbit/s.

a) What is the maximum arrival rate of low priority packets for the system to remain stable?
b) What is the waiting time of high-priority packets in case (a) and in the case when there are no low-priority packets in the network?
c) What is the waiting time of the low priority packets if the arrival rate is half of the maximum calculated in (a)?

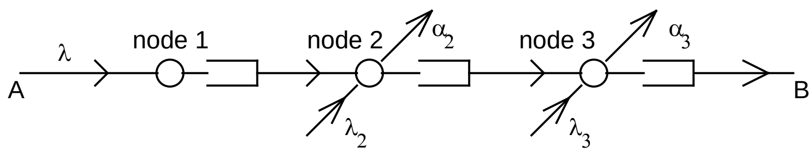
11.5. Consider a job processing system with a single server. Two types of jobs arrive at the system with two independent Poisson processes of intensity 60 jobs/s and 40 jobs/s, respectively. The processing times of the two types of jobs are exponentially distributed with mean of 12 ms and 6 ms, respectively. The system starts an automatic maintenance when it becomes idle, during which the server cannot process any jobs. The length of every maintenance period is exponentially distributed with a mean of 2 ms. Jobs need to wait in a FIFO queue with infinite size if the system is busy or in maintenance.

a) What is the mean remaining maintenance time observed by an arbitrary job when it arrives? What is the mean remaining maintenance time observed by jobs arriving when the system is empty?
b) Calculate the mean delay (waiting time + processing time) of an arbitrary job, and of the two types of jobs, respectively.

Queuing networks

12.1. Consider a connection from A to B through nodes 1,2 and 3 as shown in the figure. All the links have a capacity of C bit/s and the packet sizes are exponentially distributed with a mean of $1/\beta$ bits. Input streams 1,2 and 3 are Poissonian with intensity λ, λ_2 and λ_3. Packets leave nodes 2 and 3 through another path than to node 3 with probability $\alpha_2=\alpha_3$.

a) Calculate the utilization of the links.
b) Calculate the mean number of packets at the nodes in steady state.
c) Determine the average delay of transmitting a packet from A to B.



12.2. Consider a network of two tandem M/M/1 queues with service intensities $\mu_1=\mu_2=C\beta$.

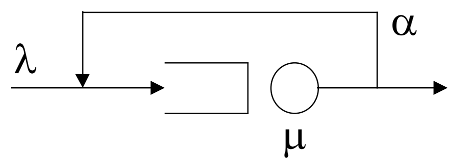
a) Give the distribution of the time a packet spends in the system in transform form and in time domain ($T^*(s)$ and $T(t)$).
b) Calculate the mean and the variance of the time a packet spends in the system.
c) Calculate the probability that the time a packet spends in the system is higher than *t*.

12.3. Consider a network of three nodes. The first and the second can be considered as M/M/1 servers, the third as an M/M/2/2 loss system. Customers arrive to the first node with intensity λ. The mean service time at the first server is x_1. From node1 customers leave to node 2 with probability β and to node 3 with probability $1-\beta$. Mean service times at nodes 2 and 3 are x_2 and x_3 respectively. Consider $\lambda=4$ customers per minute, $x_1=10$sec, $x_2=30$sec, $x_3=20$sec, $\beta=0.2$.

a) Calculate the arrival rates at nodes 2 and 3.
b) Determine the mean number of customers at nodes 1 and 2 in steady state.
c) Calculate the number of customers rejected per minute at node 3.
d) Calculate the mean time spent in the system of a lucky customer not rejected at node 3.
e) Calculate the mean waiting time of a lucky customer not rejected at node 3.

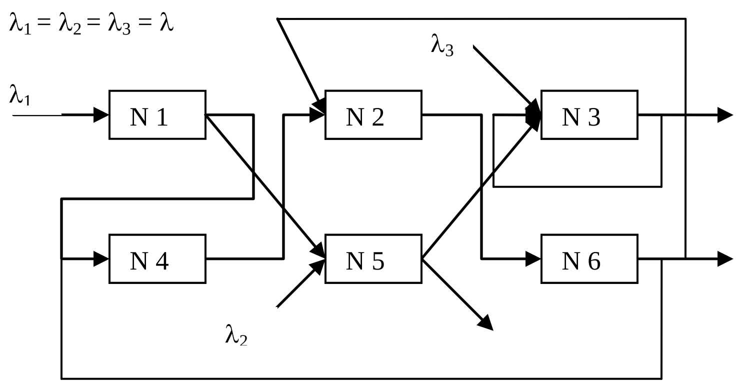
12.4. In a round-robin multiprocessing system customers are not served in one turn, but after a short service time they are re-queued at the server for further processing. The round-robin system can be modeled with the following queuing system, where α is the probability that a customer needs further processing, μ is the mean service time a customer receives in one turn and λ is the arrival rate. (We assume that the service time slots are exponentially distributed.)

a) Give the distribution of the full service time of a customer.
b) Calculate the average service time of a customer.
c) Draw the state transition matrix and give the distribution function of the number of customers in the system.
d) Calculate the mean number of customers in the system.
e) Calculate the average waiting time of a customer.



12.5. Consider the queuing network in the figure below. Traffic arrives to the network at average rate λ per input (Poisson); all service rates are identically and exponentially distributed with average rate μ. Traffic is evenly spread over all outputs from a node. All nodes have unlimited waiting room.

a) Find the minimal mean service rate $\mu = k \times \lambda$, where k is an integer, so that all nodes have a finite expected number of customers.
b) Find the state probabilities for the network.
c) Omit the in and out flows for the network and find the state probabilities for the resulting closed queuing network. There are three customers in the system, and we assume that none of them is in Node 3.



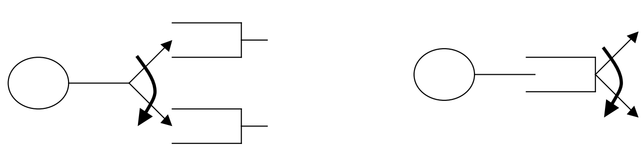
Collection of exam problems

1. Osquar works as an independent consultant. His current project is to dimension a queuing system in a router. The specification prescribes a loss probability of one-tenth of a percent when the arrival and service rates are 36 thousand packets per second and 40 thousand packets per second, respectively. Osquar assumes Poissonian arrival and service processes and computes the size from an M/M/1 system as the probability of having K+1 packets in the system (i.e., $P(\text{loss}) \approx P(N=K+1)$).
   a) Calculate the needed buffer size according to Osquar's M/M/1 approximation.
   b) Calculate the buffer size for the corresponding M/M/1/K system with finite queue. Compare it with Osquar's approximation. In general does his approximation lead to an over or an under dimensioning of the buffer size (for any load and any prescribed loss of probability)?

2. A web server holds files with sizes distributed according to $P(X < x) = 1-(a/x)^b$, where $a, b > 0$ and $x > a$. Let $a = 250$ bytes. Requests for files can be modeled as a Poisson process with 1500 requests per second on the average. All files are equally likely to be requested. The limitation in downloading from the server is its network connection, and not disk access. The files that had been fetched from disk await transmission in an infinite network access buffer. The access link has a capacity of 8 Mbit/s.
   a) What is the expected utilization of the output link from the server for $b > 1$?
   b) Find the average waiting time in the buffer as a function of b and plot it for $b < 2$.
   c) Suppose the requested files are sorted in two separate buffers. All files smaller than the median are put in one buffer and those that are longer will go to the other buffer. The buffer for short files is given non-preemptive high priority over the other buffer. Find the waiting times in the two buffers for $b = 3$.

3. Compare an M/M/∞ system with a queuing system with one server and infinite queuing capacity that can be modeled as a birth-death process with $\lambda_k=\lambda$ (k=0,1,2,…) and $\mu_k=k\mu$ (k=0,1,2,…). For these two systems determine and compare:
   a) State probabilities.
   b) Average number of customers in the system.
   c) Mean time in the system.
   d) Mean waiting time.
   e) Probability that a customer has to wait.
   f) Offered load.

4. Traffic dispersion is a technique by which traffic from a source is spread evenly over a number of paths that lead to the destination (see the figure below). Suppose a source can be modeled as a Poisson process, generating 100 packets per second. The packets have exponentially distributed length with an average of 200 bytes.
   a) Which strategy is more efficient: to spread the packets randomly over all paths, or to spread them cyclically round robin? Motivate your answer.
   b) Compare the system time for the case when packets are randomly dispersed with an undispersed system, in which all packets enter a queue that feeds a link of 200 kbit/s. Assume that the dispersion spreads the packets over 5 paths and each path has a capacity of 40 kbit/s. The buffers are infinite.
   c) Compare the two cases in (b) with a system that buffers the packets before dispersing them over 5 links of 40 kbit/s capacity each (see the figure below).



5. One often assumes Poisson arrivals and exponentially distributed packet lengths when calculating queue length and waiting times. For this problem the average arrival rate is 800 packets per second with exponentially distributed, independent interarrival times. The packet length has an average of 250 byte in all cases.
   a) How does the average queue length change from the M/M/1 case if the exponential packet length distribution is truncated at 500 bytes? The output link has a capacity of 2Mbit/s.
   b) How does the average queue length change if the packet lengths instead are evenly distributed over an interval starting at 0?

6. Consider an $M/E_2/1/2$ queuing system with λ=1 job/minute and μ=1.5 job/minute.
   a) Give the state transition diagram and calculate the state probabilities.
   b) Calculate the blocking probability, the mean time the system remains in blocking state and the mean time the system is in non-blocking state.
   c) Calculate the mean time a job spends in the system.

7. Consider a messaging service with two answering machines and one buffer position. Calls arrive to this system in a Poissonian fashion with arrival rate of 60 calls per minute. Assume exponentially distributed call holding times with a mean of 2 seconds.
   a) Draw the state transition diagram.
   b) Determine the probability that a customer has to wait.
   c) Determine the mean waiting time.
   d) Calculate the mean blocking time.

8. Consider two M/M/2 loss systems, A and B. A has 40 subscribers; to B there are 4 subscribers connected. Each subscriber generates on the average 1 call per minute. The mean service time is 6 seconds.
   a) Compare the call blocking probability for A and B.
   b) Compare the mean blocking time for A and B.
   c) Compare the mean time without blocking for A and B.
   d) How many additional servers do you have to provide to system A to achieve a time blocking probability that is not higher than that of system B?

9. The measured packet length distribution in the Internet has been approximated with the following: P(length = 40 byte) = 0.65, P(length = 595 byte) = 0.20 and P(length = 1500 byte) = 0.15. Assume that packets arrive to a queue according to a Poisson process with an average arrival rate of 10^4 packets per second. The queue feeds a link with capacity 34 Mbit/s.
   a) Find the average waiting time in the system.
   b) Give the *z*-transform of the distribution of the number of packets in the system.
   c) What would the waiting time be if an exponential distribution were fitted to the measured packet-length distribution so that the expected packet length is the same as in the distribution above? Compare the result from (a) and explain.

10. Consider an M/M/1 system with limited buffer and an offered load of ρ = 0.5.
    a) How many queuing positions, K, are required to achieve a blocking probability of less than 6.25%?
    b) Determine the probability that there are more than K+1 customers in an M/M/1 system with unlimited buffer and the same offered load (i.e., 0.5). (Assume K = 5 if you did not solve part a)).