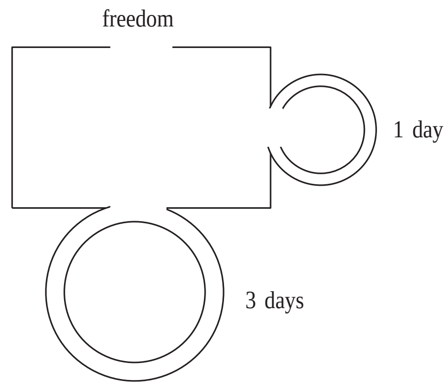
1. Repetition – probability theory and transforms

Ex. 1.1.

$\bar{S}$ – time to get out of the prison

$E(\bar{S}) =?$



$$E(\bar{S}) = \sum_{i=1}^{3} E(\bar{S}|\text{door } i)P(\text{door } i)$$

$$P(\text{door } i) = \frac{1}{3}$$

$$E(\bar{S}|\text{door } 1) = 0$$

$$E(\bar{S}|\text{door } 2) = 1 + E(\bar{S})$$

$$E(\bar{S}|\text{door } 3) = 3 + E(\bar{S})$$

$$E(\bar{S}) = \frac{1}{3}(1+E(\bar{S}))+\frac{1}{3}(3+E(\bar{S})) = \frac{4}{3}+\frac{2}{3}E(\bar{S})$$

$$E(\bar{S}) = 4$$

Ex. 1.2.

$$\lambda_1 = 0.1,\ \lambda_2 = 0.02,\ a = 0.2\, b = 0.8$$

$$a\frac{1}{\lambda_1} + b\frac{1}{\lambda_2} = 42$$

$$a\frac{2}{\lambda_1^2} + b\frac{2}{\lambda_2^2} = 4040$$

$$a \cdot d_1 + b \cdot d_2 = 42$$

$$a \cdot d_1^2 + b \cdot d_2^2 = 4040$$

$$0.2 \cdot d_1 + 0.8 \cdot d_2 = 42$$

$$0.2 \cdot d_1^2 + 0.8 \cdot d_2^2 = 4040$$

$$0.2 \left(\frac{42 - 0.8 \cdot d_2}{0.2}\right)^2 + 0.8 \cdot d_2^2 = 4040$$

$$d_2^2 - 84 \cdot d_2 + 1195 = 0 \qquad \Rightarrow \qquad d_{2/1} = 65.85, \qquad \underline{d_{2/2} = 18.15}$$

$$d_1 = \frac{42 - 0.8 \cdot d_2}{0.2} = \frac{42 - 0.8 \cdot 18.15}{0.2} = 137.41$$

Ex. 1.3.

a)

$$\sum_{k=0}^{\infty} P_k = \sum_{k=0}^{\infty} \frac{a^k}{k!}e^{-a} = e^{-a}\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^{-a}e^{a} = 1$$

b)

$$P(z) = \sum_{k=0}^{\infty} z^k \frac{a^k}{k!}e^{-a} = e^{-a}\sum_{k=0}^{\infty} \frac{(za)^k}{k!} = e^{-a}e^{az} = e^{-a(1-z)}$$

c)

$$E\{X(X-1)\cdot \ldots \cdot (X-r+1)\} = \sum_{k=0}^{\infty} k(k-1)\cdot \ldots \cdot (k-r+1)P_k =$$

$$= \sum_{k=0}^{\infty} k(k-1)\cdot\ldots\cdot(k-r+1)\frac{a^k}{k!}e^{-a} = a^r e^{-a}\sum_{k=0}^{\infty}\frac{a^{(k-r)}}{(k-r)!} = a^r e^{-a} e^a = a^r \qquad (r=1,2,\ldots)$$

$$E(X) = a$$

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = E\{X(X-1)\} + E(X) - E(X)^2 = a^2 + a - a^2 = a$$

$$\frac{\mathrm{d}^r P(z)}{\mathrm{d}z^r} = \sum_{k=0}^{\infty} k(k-1)\cdot\ldots\cdot(k-r+1)P_k z^{(k-r)}$$

$$E\{X(X-1)\cdot\ldots\cdot(X-r+1)\} = \lim_{z\to 1}\frac{\mathrm{d}^r P(z)}{\mathrm{d}z^r} \qquad (r=1,2,\ldots)$$

$$\frac{\mathrm{d}^r P(z)}{\mathrm{d}z^r} = \frac{\mathrm{d}^r}{\mathrm{d}z^r}\left\{e^{-a(1-z)}\right\} = a^r e^{-a(1-z)} \qquad (r=0,1,\ldots)$$

$$\lim_{z\to 1}\frac{\mathrm{d}^r P(z)}{\mathrm{d}z^r} = a^r \qquad (r=0,1,\ldots)$$

$$E(X) = a$$

$$E\{X(X-1)\} = a^2; \quad \mathrm{Var}(X) = a; \quad E\{X(X-1)\cdot\ldots\cdot(X-r+1)\} = a^r \qquad (r=1,2,\ldots)$$

Ex. 1.4.

$$P(z) = E\left\{z^X\right\} = \sum_{k=0}^{\infty} z^k P(X=k) = E\left\{z^{X_1+X_2+\cdots+X_n}\right\} = E\left\{z^{X_1}\cdot z^{X_2}\cdot\ldots\cdot z^{X_n}\right\} =$$

$$E\left\{z^{X_1}\right\}\cdot E\left\{z^{X_2}\right\}\cdot\ldots\cdot E\left\{z^{X_n}\right\}$$

$$E\left\{z^{X_i}\right\} = \sum_{k=0}^{\infty} z^k P(X_i=k) = \sum_{k=0}^{\infty} z^k \frac{{a_i}^k}{k!}e^{-a_i} = e^{-a_i(1-z)} \qquad (r=1,2,\ldots)$$

$$P(z) = e^{-a_1(1-z)}\cdot e^{-a_2(1-z)}\cdot\ldots\cdot e^{-a_n(1-z)} = e^{-a(1-z)}$$

If $a = \sum_{i=1}^{n} a_i$, X is Poisson distributed with parameter a.

Ex. 1.5.

a)

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left\{1-e^{-ax}\right\} = \begin{cases} ae^{-ax} & x>0 \\ 0 & x\le 0 \end{cases}$$

b)

$$\overline{F}(x) = P(X>x) = 1 - P(X\le x) = 1 - F(x)$$

$$\overline{F}(x) = \begin{cases} 1 & x<0 \\ ae^{-ax} & x\ge 0 \end{cases}$$

c)

$$F^*(s) = E(e^{-sX}) = \int_0^{\infty} e^{-sx}ae^{-ax}\mathrm{d}x = \frac{a}{s+a} \qquad (Re(s) > -a)$$

d)

$$E\{X^0\} = \int_0^{\infty} f(x)\mathrm{d}x = \int_0^{\infty} ae^{-ax}\mathrm{d}x = -e^{-ax}\Big|_0^{\infty} = 0+1 = 1$$

$$E\{X^k\} = \int_0^\infty x^k a e^{-ax} \mathrm{d}x = (x^k) - e^{-ax}\Big|_0^\infty + k\int_0^\infty x^{k-1} e^{-ax}\mathrm{d}x =$$

$$= 0 + 0 + \frac{k}{a} EX^{k-1} \qquad (k = 1, 2, \ldots)$$

$$E\{X^k\} = \frac{k}{a} \cdot \frac{k-1}{a} \cdot \ldots \cdot \frac{1}{a} = \frac{k!}{a^k} \qquad (k = 0, 1, \ldots)$$

$$E\{X\} = \frac{1}{a}; \quad \sigma^2 = \mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{2}{a^2} - \frac{1}{a^2} = \frac{1}{a^2}$$

$$\sigma = \sqrt{\mathrm{Var}(X)} = \frac{1}{a}; \quad C = \frac{\sigma}{m} = 1$$

$$F^*(s) = E(e^{-sX}) = \int_0^\infty e^{-sx} f(x)\mathrm{d}x \quad \Rightarrow \quad \frac{\mathrm{d}^k F^*(s)}{\mathrm{d}s^k} = (-1)^k \int_0^\infty x^k e^{-sx} f(x)\mathrm{d}x$$

$$E\{X^k\} = (-1)^k \lim_{s\to 0} \frac{\mathrm{d}^k F^*(s)}{\mathrm{d}s^k} \qquad (k = 0, 1, \ldots)$$

$$\frac{\mathrm{d}^k F^*(s)}{\mathrm{d}s^k} = \frac{\mathrm{d}^k}{\mathrm{d}s^k}\left\{\frac{a}{s+a}\right\} = \frac{(-1)^k a k!}{(s+a)^{k+1}} \qquad (k = 0, 1, \ldots)$$

$$E\{X^k\} = (-1)^k \cdot (-1)^k \cdot \frac{k!}{a^k} = \frac{k!}{a^k} \qquad (k = 0, 1, \ldots)$$

$$E\{X\} = \frac{1}{a}; \quad \sigma^2 = \mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{1}{a^2}$$

$$\sigma = \sqrt{\mathrm{Var}(X)} = \frac{1}{a}; \quad C = \frac{\sigma}{m} = 1$$

Ex. 1.6.

a)

$$\overline{F}_X(x) = P(X > x) = P\{X_1, X_2, \ldots, X_n > x\} = P\{X_1 > x, X_2 > x, \ldots, X_n > x\} =$$

$$= \{\text{because } X_i \text{ are independent}\} = P\{X_1 > x\} \cdot P\{X_2 > x\} \cdot \ldots \cdot P\{X_n > x\} =$$

$$= \prod_{i=1}^n \overline{F}_{X_i}(x) = \prod_{i=1}^n e^{-ax} = e^{-nax} \Rightarrow X \text{ is exponentially distributed with mean } \frac{1}{na}$$

$$F_X(x) = P(X \le x) = 1 - e^{-nax} \qquad (x \ge 0)$$

b)

$$F_X(x) = P(X \le x) = P\{\max(X_1, X_2, \ldots, X_n) \le x\} = P\{X_1 \le x, X_2 \le x, \ldots, X_n \le x\} =$$

$$= \{\text{because } X_i \text{ are independent}\} = P\{X_1 \le x\} \cdot P\{X_2 \le x\} \cdot \ldots \cdot P\{X_n \le x\} =$$

$$= \prod_{i=1}^n F_{X_i}(x) = \left(1 - e^{-ax}\right)^n$$

$$\overline{F}_X(x) = P(X > x) = 1 - \left(1 - e^{-ax}\right)^n \qquad (x \ge 0)$$

Ex. 1.7.

a)

$$F^*(s) = E(e^{-sX}) = \int_0^\infty e^{-sx} f(x)\mathrm{d}x = E\{e^{-s(X_1 + X_2 + \cdots + X_r)}\} =$$

$$= \{\text{because } X_i \text{ are independent}\} = E\{e^{-sX_1}\} \cdot E\{e^{-sX_2}\} \cdot \ldots \cdot E\{e^{-sX_r}\} =$$

$$= \frac{a}{s+a} \cdot \frac{a}{s+a} \cdot \ldots \cdot \frac{a}{s+a} = \left(\frac{s}{s+a}\right)^r \qquad (r = 1, 2, \ldots;\ Re(s) > -a)$$

We recognize that the above expression is the Laplace transform of the Erlang-r distribution:

$$f(x) = a \cdot \frac{(ax)^{r-1}}{(r-1)!} \cdot e^{-ax} \qquad (x > 0;\ r = 1, 2, \ldots)$$

$$\overline{F}(x) = \int_x^\infty f(u)\mathrm{d}u = \int_x^\infty a\frac{(an)^{r-1}}{(r-1)!}e^{-an}\mathrm{d}u =$$

$$= \Big| \left(\frac{(an)^{r-1}}{(r-1)!}\right)\left(-e^{-an}\right)\Big|_x^\infty + \int_x^\infty a\frac{(an)^{r-2}}{(r-2)!}e^{-an}\mathrm{d}u =$$

$$= \frac{(ax)^{r-1}}{(r-1)!}e^{-ax} + \Big| \left(\frac{(an)^{r-2}}{(r-2)!}\right)\left(-e^{-an}\right)\Big|_x^\infty + \int_x^\infty a\frac{(an)^{r-3}}{(r-3)!}e^{-an}\mathrm{d}u =$$

$$= \frac{(ax)^{r-1}}{(r-1)!}e^{-ax} + \frac{(ax)^{r-2}}{(r-2)!}e^{-ax} + \cdots + \frac{(ax)^1}{1!}e^{-ax} + \int_x^\infty ae^{-an}\mathrm{d}u =$$

$$= \sum_{j=0}^{r-1}\frac{(ax)^j}{j!}e^{-ax} \qquad (x \le 0)$$

$$F(x) = 1 - \overline{F}(x) = 1 - \sum_{j=0}^{r-1}\frac{(ax)^j}{j!}e^{-ax} = \sum_{j=r}^{\infty}\frac{(ax)^j}{j!}e^{-ax} \qquad (x \le 0)$$

$$m = E\{X\} = -\lim_{s\to 0}\frac{\mathrm{d}F^*(s)}{\mathrm{d}s} = -\lim_{s\to 0}\left\{r\left(\frac{a}{s+a}\right)^{r-1}\cdot\frac{-a}{(s+a)^2}\right\} = \frac{r}{a}$$

$$E\{X^2\} = \lim_{s\to 0}\frac{\mathrm{d}^2F^*(s)}{\mathrm{d}s^2} = \lim_{s\to 0}\left\{\frac{\mathrm{d}}{\mathrm{d}s}\left[-\frac{r}{a}\left(\frac{a}{s+a}\right)^{r+1}\right]\right\} =$$

$$= \lim_{s\to 0}\left\{\frac{r(r+1)}{a^2}\left(\frac{a}{s+a}\right)^r\right\} = \frac{r(r+1)}{a^2}$$

$$\sigma^2 = \mathrm{Var}(X) = \frac{r(r+1)}{a^2} - \frac{r^2}{a^2} = \frac{r}{a^2} \quad \Rightarrow \quad \sigma = \frac{\sqrt{r}}{a}$$

$$C = \frac{\sigma}{m} = \frac{1}{\sqrt{r}} \qquad (r = 1, 2, \ldots)$$

Alternative method:

$$m = E(X) = \int_0^\infty xf(x)\mathrm{d}x = \frac{r}{a}\int_0^\infty a\frac{(ax)^r}{r!}e^{-ax}\mathrm{d}x = \frac{r}{a}\cdot 1 = \frac{r}{a}$$

$$E(X^2) = \int_0^\infty x^2 f(x)\mathrm{d}x = \frac{(r+1)r}{a^2}\int_0^\infty a\frac{(ax)^{r+1}}{(r+1)!}e^{-ax}\mathrm{d}x = \frac{(r+1)r}{a^2}\cdot 1 = \frac{(r+1)r}{a^2}$$

$$\sigma^2 = \frac{r}{a^2};\ \sigma = \frac{\sqrt{r}}{a};\ C = \frac{1}{\sqrt{r}}$$

Ex. 1.8.

The memoryless property of the exponential distribution

$$P(\tilde{t} \le t) = 1 - e^{-\lambda t}, \qquad t \ge 0$$

$$P(\tilde{t} \le t + t_0 | \tilde{t} > t_0) = \frac{P[t_0 < \tilde{t} \le t + t_0]}{P[\tilde{t} > t_0]} = \frac{P[\tilde{t} \le t + t_0] - P[\tilde{t} \le t_0]}{P[\tilde{t} > t_0]} =$$

$$= \frac{1 - e^{-\lambda(t+t_0)} - (1 - e^{-\lambda t_0})}{1 - (1 - e^{-\lambda t_0})} = 1 - e^{-\lambda t}$$

2. Poisson process

Ex. 2.1.

Poisson arrival process ⇒ exponential inter-arrival times

$$\text{Poisson:} \quad P_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \qquad k \geq 0,\ t \geq 0$$

$$\text{Inter-arrival time } \tau: \quad P(\tau \leq t) = 1 - P(\tau > t) = 1 - P_0(t) = 1 - e^{-\lambda t}$$

Ex. 2.2.

Multiplexer. The inter-arrival time for the combined flow:

$$P(\tau \leq t) = 1 - \prod_{i=1}^{k} P(\tau_i > t) = 1 - \prod_{i=1}^{k} e^{-\lambda_i t} = 1 - e^{\sum_{i=1}^{k} \lambda_i t}$$

⇒ the inter-arrival time is exponential with $\lambda = \sum_{i=1}^{k} \lambda_i$

⇒ the arrival process is Poisson with λ

Ex. 2.3.

$A_i(t) = 1 - P(\tilde{t}_i > t)$; $\quad \tilde{t}_i$ = inter-arrival time for stream i

$P(\tilde{t}_i > t) = P(0 \text{ arrivals in main stream in } t)+$

$+\sum_{k=1}^{\infty} P(k \text{ arrivals in main stream and substream } i \text{ not picked})$

$P_{ni} = P(n \text{ arrivals in main stream in time } t \text{ and substream } i \text{ not picked})$

$$P_{ni} = \frac{(\lambda t)^n}{n!} e^{-\lambda t}(1-p_i)^n, \quad P(\tilde{t}_i > t) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} (1-p_i)^n$$

$$P(\tilde{t}_i > t) = e^{-\lambda p_i t}; \quad A_i(t) = 1 - e^{-\lambda p_i t}; \quad a_i(t) = \lambda p_i e^{-\lambda p_i t}$$

This is a Poisson process with parameter λp_i!

Ex. 2.4.

$\lambda = 1000 \frac{packets}{sec}$

a)

$$P_0(t) = e^{-\lambda t}; \qquad P_0(10^{-3}) = \frac{1}{e}$$

b)

$$F(s) = \left(\frac{\lambda}{s+\lambda}\right)^{500} \qquad \text{(500 stage Erlang)}$$

$$E(\overline{T}) = -\frac{\mathrm{d}}{\mathrm{d}s} F_x(s)\big|_{s=0} = -\lambda^{500}(-500)\left(\frac{1}{s+\lambda}\right)\big|_{s=0}^{501} = \frac{500}{\lambda}$$

$$E(\overline{T}^2) = \frac{\mathrm{d}^2}{\mathrm{d}s^2} F_x(s)\big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s} \lambda^{500}(-500)\left(\frac{1}{s+\lambda}\right)^{501}\Big|_{s=0} = \frac{500 \cdot 501}{\lambda^2}; \qquad \mathrm{Var}(\overline{T}) = \frac{500}{\lambda^2}$$

Ex. 2.5.

$$P(S_N = k|N = n) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & n \geq k \quad \text{(binomial probability law)} \\ 0 & n < k \end{cases}$$

$$P(S_N = k) = \sum_{n=k}^{\infty} P(S_N = k|N = n)P(N = n); \qquad P(N = n) = \frac{\lambda^n}{n!} e^{-\lambda}$$

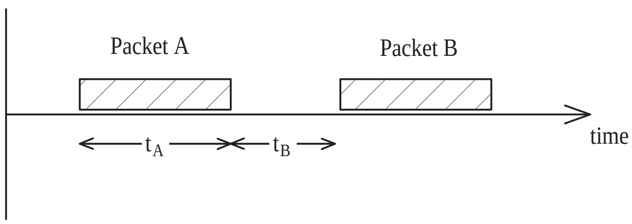
$$P(S_N = k) = (p\lambda)^k e^{-\lambda p} \qquad \text{(a Poisson process with parameter } \lambda p)$$

Ex. 2.6.

For simplicity, let us assume that the packet transmission time is unity (one unit of measure). Let us further assume that the number of nodes is very large and can be taken as infinity, then the new packet arrivals per unity time is a Poisson random variable with rate λ. If the collisions of packets, and hence retransmissions are fairly random, we can approximate the combined arrival process of the retransmissions and new packets generated by other nodes as a Poisson process with parameter $G > \lambda$. Hence, the throughput is given by

$$S = G \cdot P[\text{a sucessuful transmission}] = G \cdot P_{succ}$$

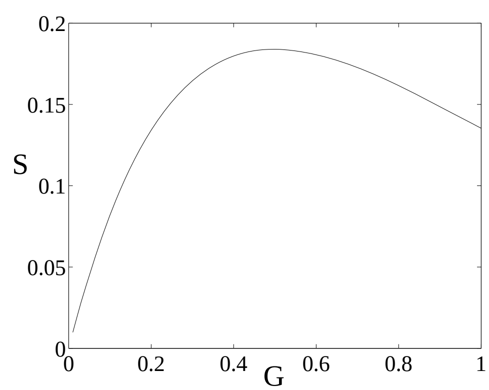
Now, it remains for us to calculate the probability of a successful transmission. If we examine the figure, we see that a packet that is generated by other nodes during $t_A = 1$ will collide with packet A. And the time gap t_g between two packets has to be at least equal to the packet transmission time for a packet generated during t_g not to collide with packet B.



Therefore, to have a successful transmission, there should be no packet generated within 2 time units. The probability of this is given by

$$P_{succ} = \frac{G^0}{0!} e^{-2G} = e^{-2G}$$

Therefore $\quad S = Ge^{-2G}$



Ex. 2.7.

$A_k(t)$: arriving customer at time t sees k customers in the queue

$P_k(t)$: state probability

$$A_k(t) = P(N(t) = k|\text{ arrival at time } t) = \frac{P(N(t) = k, \text{ arrival at time } t)}{P(\text{arrival at time } t)} =$$

$$= \frac{P(\text{arrival at time } t|N(t) = k) \cdot P(N(t) = k)}{P(\text{arrival at time } t)}$$

Since the interarrival time is memoryless, we know that the arrivals are independent of $N(t)$, so that:

$$P(\text{arrival at time } t | N(t) = k) = P(\text{arrival at time } t)$$

Hence:

$$A_k(t) = P(N(t) = k) = P_k(t)$$

When $t \to \infty$: $A_k = P_k$.

3. Balance equations, birth-death processes and continuous time Markov chains

Ex. 3.1.

Pure birth process

$$\frac{\mathrm{d}P_k(t)}{\mathrm{d}t} = -\lambda P_k(t) + \lambda P_{k-1}(t), \qquad k \geq 1$$

$$\frac{\mathrm{d}P_0(t)}{\mathrm{d}t} = -\lambda P_0(t), \qquad k = 0$$

$$P_k(0) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

$$P_0(t) = e^{-\lambda t}$$

$$\frac{\mathrm{d}P_1(t)}{\mathrm{d}t} = -\lambda P_1(t) + \lambda e^{-\lambda t} \Rightarrow P_1(t) = \lambda t e^{-\lambda t}$$

$$P_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \qquad k \geq 0,\ t \geq 0 \quad \Rightarrow \quad \text{Poisson distribution!}$$

Ex. 3.2.

Suppose that we just arrived to a state. Time until the next birth, X, is exponentially distributed with distribution function $F_X(t) = 1 - e^{-\lambda t}$ and time to the next death, Y, is also exponentially distributed with distribution function $F_Y(t) = 1 - e^{-\mu t}$. If Z is the time that we spend in a state, then $Z = min(X, Y)$. If $F_Z(t)$ is the distribution function for Z, we get:

$$F_Z(t) = P(Z \leq t) = P(min(X,Y) \leq t) = 1 - P(min(X,Y) > t) = 1 - P(X > t, Y > t) =$$
$$1 - P(X > t)P(Y > t) = 1 - (1 - P(X \leq t))(1 - P(Y \leq t)) = 1 - (1 - F_X(t))(1 - F_Y(t)) =$$
$$= 1 - e^{-\lambda t} e^{-\mu t} = 1 - e^{-(\lambda+\mu)t}$$

which is the distribution function of an exponentially distributed r.v. with mean $\frac{1}{\mu+\lambda}$.

Ex. 3.3.

T: conversation time

$$P(T \leq t) = 1 - e^{-\mu t}, \qquad f_T(t) = \mu e^{-\mu t}$$

X: number of calls

$$P(X = k | T = t) = P(\text{arriving calls during a conversation of length } t = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

$$E(X|T = t) = \sum_{k=1}^{\infty} k \frac{(\lambda t)^k}{k!} e^{-\lambda t} = \lambda t e^{-\lambda t} \sum_{k=1}^{\infty} \frac{(\lambda t)^{k-1}}{(k-1)!} = \lambda t e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} = \lambda t$$

$$E(X) = \int_{t=0}^{\infty} E(X|T = t) f_T(t) \mathrm{d}t = \int_{t=0}^{\infty} \lambda t \mu e^{-\mu t} \mathrm{d}t = \lambda t e^{-\mu t}\Big|_0^{\infty} + \int_{t=0}^{\infty} \lambda e^{-\mu t} \mathrm{d}t =$$
$$= -\frac{\lambda}{\mu} e^{-\mu t}\Big|_0^{\infty} = \frac{\lambda}{\mu}$$

Ex. 3.4.

X_1: size of message type 1

X_2: size of message type 2

S: service time

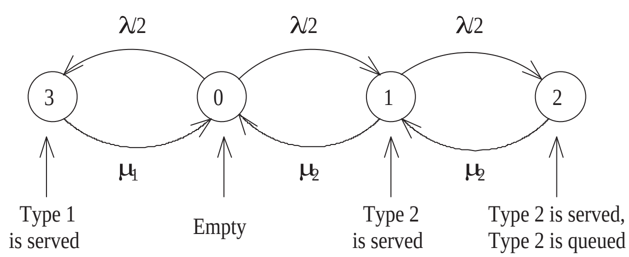
$$\overline{X}_1 = 300\frac{bits}{msg}, \qquad \overline{X}_2 = 150\frac{bits}{msg}, \qquad \overline{S}_1 = \frac{\overline{X}_1}{R}, \qquad \overline{S}_2 = \frac{\overline{X}_2}{R}$$

a)

$$\overline{S} = E(S) = E(S|\text{msg type 1}) \cdot P(\text{msg type 1}) + E(S|\text{msg type 2}) \cdot P(\text{msg type 2}) =$$
$$= \frac{300}{4800} \cdot \frac{1}{2} + \frac{150}{4800} \cdot \frac{1}{2} \approx 47ms$$
$$C_S^2 = \frac{\sigma_S^2}{\overline{S}^2} = \frac{\overline{S^2}}{\overline{S}^2} - 1$$
$$\overline{S^2} = E(S^2) = E(S^2|\text{msg type 1}) \cdot P(\text{msg type 1}) + E(S^2|\text{msg type 2}) \cdot P(\text{msg type 2}) =$$
$$= \frac{1}{2}\left[2\left(\frac{300}{4800}\right)^2 + 2\left(\frac{150}{4800}\right)^2\right] = \frac{5}{32^2}s^2$$
$$C_S^2 = \frac{20}{9} \quad \Rightarrow \quad C_S = \frac{\sqrt{20}}{3}$$

b)

$$\mu_1 = 16\frac{msg}{sec}, \qquad \mu_2 = 32\frac{msg}{sec}$$



$$\lambda = 10, \quad \mu_1 = 16, \quad \mu_2 = 32$$

(1) $P_3 = \frac{\lambda}{2\mu_1}P_0$

(2) $P_1 = \frac{\lambda}{2\mu_2}P_0$

(3) $P_2 = \frac{\lambda^2}{4\mu_2^2}P_0$

(4) $P_0 + P_1 + P_2 + P_3 = 1$

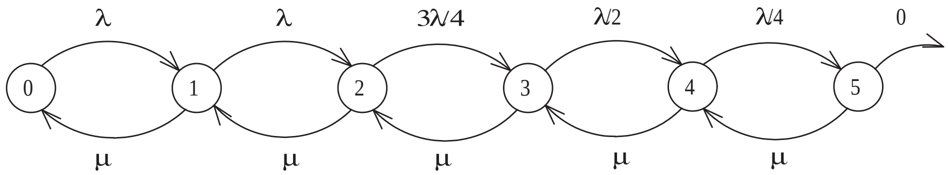
$$P_0 \approx 0.670, \quad P_1 \approx 0.105, \quad P_2 \approx 0.016, \quad P_3 \approx 0.209$$
$$E(T_1) = \frac{1}{\mu_1} = 62.5ms$$
$$E(T_2) = \frac{1}{P_0 + P_1}\left(P_0 \cdot \frac{1}{\mu_2} + P_1\left(\frac{1}{\mu_2} + \frac{1}{\mu_2}\right)\right) = 35.5ms$$

c)

$$Pr(\text{loss type 1}) = Pr(\text{not in state 0}) = 1 - 0.67 \approx 0.33$$
$$Pr(\text{loss type 2}) = Pr(\text{in state 2 or 3}) = P_2 + P_3 \approx 0.225$$

Ex. 3.5.

a)



$$\lambda = \frac{1}{7}\frac{job}{sec}, \quad \mu = \frac{1}{6}\frac{job}{sec}, \quad \rho = \frac{\lambda}{\mu} = \frac{6}{7}$$

(1) $\lambda P_0 = \mu P_1 \quad \Rightarrow \quad P_1 = \rho P_0$

(2) $\lambda P_1 = \mu P_2 \quad \Rightarrow \quad P_2 = \rho P_1 = \rho^2 P_0$

(3) $\frac{3}{4}\lambda P_2 = \mu P_3 \quad \Rightarrow \quad P_3 = \frac{3}{4}\rho P_2 = \frac{3}{4}\rho^3 P_0$

(4) $\frac{1}{2}\lambda P_3 = \mu P_4 \quad \Rightarrow \quad P_4 = \frac{1}{2}\rho P_3 = \frac{3}{8}\rho^4 P_0$

(5) $\frac{1}{4}\lambda P_4 = \mu P_5 \quad \Rightarrow \quad P_5 = \frac{3}{32}\rho^5 P_0$

(6) $\sum_{i=0}^{5} P_i = 1$

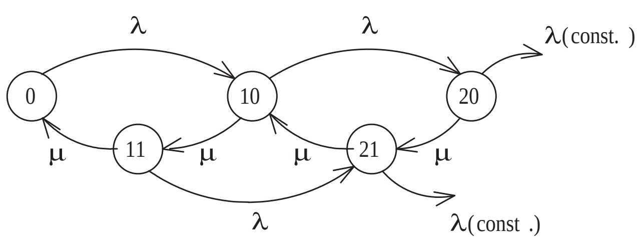
$$P_0 = \frac{1}{1 + \rho + \rho^2 + \frac{3}{4}\rho^3 + \frac{3}{8}\rho^4 + \frac{3}{32}\rho^5} \approx 0.30$$

$$N = 0.3 \cdot (\rho + 2\rho^2 + \frac{3 \cdot 3}{4}\rho^3 + \frac{4 \cdot 3}{8}\rho^4 + \frac{5 \cdot 3}{32}\rho^5) \approx 1.43$$

b)

Jobs served in 100 sec: $\mu \cdot (1 - P_0) \cdot 100 \approx 11.66$

Ex. 3.6.



$$\rho = \lambda_{eff}\bar{x}, \quad \bar{x} = E(x_1) + E(x_2) = 60ms, \quad \lambda_{eff} = \frac{\rho}{\bar{x}} = \frac{1 - P_0}{\bar{x}} = \frac{0.4}{60}\frac{1}{ms}, \quad \mu = \frac{1}{30}\frac{1}{\mu s}$$

(1) $P_0 = 1 - \rho = 0.6$

(2) $P_0 \lambda = P_{11}\mu$

(3) $P_{10}(\lambda + \mu) = P_0\lambda + P_{21}\mu$

(4) $P_{11}(\lambda + \mu) = P_{10}\mu$

(5) $P_{20}\mu = P_{10}\lambda$

(6) $P_{21}\mu = P_{11}\lambda + P_{20}\mu$

(7) $P_0 + P_{10} + P_{11} + P_{20} + P_{21} = 1$

$$P_{10} = \frac{\lambda}{\mu}\left(1 + \frac{\lambda}{\mu}\right) P_0$$

$$P_{11} = \frac{\lambda}{\mu} P_0$$

$$P_{20} = \left(\frac{\lambda}{\mu}\right)^2 \left(1 + \frac{\lambda}{\mu}\right) P_0$$

$$P_{21} = \left(\frac{\lambda}{\mu}\right)^2 P_0 + \left(\frac{\lambda}{\mu}\right)^2 \left(1 + \frac{\lambda}{\mu}\right) P_0$$

$$P_0 = \frac{1}{1 + 2\frac{\lambda}{\mu} + 4\left(\frac{\lambda}{\mu}\right)^2 + 2\left(\frac{\lambda}{\mu}\right)^3} = 0.6 \quad \Rightarrow \quad \frac{\lambda}{\mu} \approx 0.2229$$

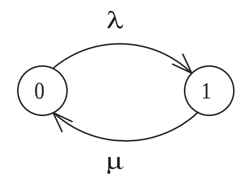
$$P_{10} = 0.1636, \quad P_{11} = 0.1337, \quad P_{20} = 0.0365, \quad P_{21} = 0.0663$$

$$\text{Little: } T = \frac{\overline{N}}{\lambda_{eff}} = \frac{0 \cdot P_0 + 1 \cdot (P_{10} + P_{11}) + 2 \cdot (P_{20} + P_{21})}{\rho/\bar{x}} \approx 75.4ms$$

Ex. 3.7.

a)

$$P_0 \cdot \frac{\lambda}{\mu} = P_1 \quad \Rightarrow \quad P_0 = \frac{P_0}{2} \quad \Rightarrow \quad P_0 = \frac{1}{3}, \qquad P_1 = \frac{2}{3}$$



Number of arrivals in time t is $P_0 \cdot a_0 + P_1 \cdot a_1$

b)

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

$$P(t) = P(0) \cdot e^{Qt}$$

$$e^{Qt} = I + Qt + \frac{Q^2t^2}{2!} + \frac{Q^3t^3}{3!} + \cdots = \begin{bmatrix} \frac{\mu+\lambda e^{-(\lambda+\mu)t}}{\lambda+\mu} & -\frac{\lambda(e^{-(\lambda+\mu)t}-1)}{\lambda+\mu} \\ -\frac{\mu(e^{-(\lambda+\mu)t}-1)}{\lambda+\mu} & \frac{\lambda+\mu e^{-(\lambda+\mu)t}}{\lambda+\mu} \end{bmatrix}$$

$$e^{Qt}\Big|_{\lambda=2\mu} = \begin{bmatrix} \frac{1}{3}(1+2e^{-3\mu t}) & \frac{2}{3}(1-e^{-3\mu t}) \\ \frac{1}{3}(1-e^{-3\mu t}) & \frac{1}{3}(2+e^{-3\mu t}) \end{bmatrix}$$

$$P(t) = [1 \quad 0] \cdot e^{Qt}\Big|_{\mu=1} = \left[\frac{1}{3}(1+2e^{-3t}) \quad \frac{2}{3}(1-e^{-3t})\right]$$

4. Queuing systems

Ex. 4.1.

Kendall notation: A/B/X/Y/Z
A: arrival process
B: service process
X: number of servers
Y: system capacity (servers + buffers)
Z: customer population

a)
M/M/1

b)
M/D/1/1+N

c)
M/G/1/1

Ex. 4.2.

a)
M/M/C/C

b)
M/M/C/C+c

Ex. 4.3.

There are only 5 customers. They cannot utilize 10 servers and 2 buffers.

Ex. 4.4.

See 4.3.

Ex. 4.5.

a)
Offered load = Arrival rate × Service time = $300 \cdot \frac{2}{60} \cdot 4.5 = 45$ Erlang.

b)

$$\text{Utilization} = \frac{\text{Offered load}}{\text{Number of servers}} = \frac{45}{90} = 0.5$$

Ex. 4.6.

We use Little's theorem to solve the problem.

a)

$$W = \frac{N_q}{\lambda} = \frac{60}{30} = 2 \text{ s}$$

$$T = W + \bar{x} = 2 + 0.25 = 2.25 \text{ s}$$

b)

$$\lambda_{eff} = \lambda(1 - P_{loss}) = 28.5 \text{ 1/s}$$

$$N_s = \bar{x} \cdot \lambda_{eff} = 0.25 \cdot 28.5 = 7.125$$

$$N = T \cdot \lambda_{eff} = 4 \cdot 28.5 = 114$$

$$N_q = N - N_s = 114 - 7.125 = 106.875$$

$$W = \frac{N_q}{\lambda_{eff}} = \frac{106.875}{28.5} = 3.75 \text{ s}$$

5. M/M/1 queuing systems

Ex. 5.1.

The service time (transmission time in the channel) is exponentially distributed with mean $\frac{1/\mu \text{ bits}}{C \text{ bits/s}} = \frac{1}{\mu C}$ s. The load $\rho = \frac{\lambda}{\mu C}$ has to be < 1 in order for the system to be stable.
Let T be the total time in the system, W the waiting time and S the service time, we have

$$E(e^{-sT}) = E(e^{-s(W+S)}) = \{\text{independence}\} = E(e^{-sW})E(e^{-sS})$$

$$E(e^{-sS}) = \frac{\mu C}{s + \mu C}$$

$$E(e^{-sW}) = 1 - \rho + (1-\rho)\frac{\lambda}{s + \mu C - \lambda} = \frac{(1-\rho)(s + \mu c)}{s + \mu C - \lambda}$$

Therefore

$$E(e^{-sT}) = \frac{\mu C - \lambda}{s + \mu C - \lambda},$$

which means that T is exponentially distributed with mean $\frac{1}{\mu C - \lambda}$ sec.

a)

$$\frac{1}{\mu C - \lambda} \leq T_0 \quad \Leftrightarrow \quad C \geq \frac{1 + \lambda T_0}{\mu T_0} \quad \Rightarrow \quad \text{smallest value for } C \text{ is } C = \frac{1 + \lambda T_0}{\mu T_0}\frac{\text{bits}}{\text{s}}$$

Let T be the total time in the system for a message. Then $P(T > t) = e^{-(\mu C - \lambda)}t$. With the C given above, we get

$$P(T > 3T_0) = e^{-3} \approx 0.05$$

b)

$$P(T > t) = e^{-(\mu C - \lambda)t} \leq p, \quad (\mu C - \lambda)t \geq -\ln(p), \quad C \geq \frac{\lambda t - \ln(p)}{\mu t}$$

The smallest value of C is $C = \frac{\lambda t - \ln(p)}{\mu t}$

c)

$$P(T > t) = e^{-(\mu C - \lambda)t} \leq p \quad \Leftrightarrow \quad -(\mu C - \lambda)t \leq \ln(p) \quad \Leftrightarrow \quad \lambda \leq \mu C + \frac{\ln(p)}{t}$$

Hence, the biggest value of λ is $\lambda = \mu C + \frac{\ln(p)}{t}$ (which is $< \mu C$).

Ex. 5.2.

a)

$$\overline{T} = \sum_{k=0}^{\infty} E(T|k) \cdot P_k = \sum_{k=0}^{\infty} P_k \cdot (k+1)\frac{1}{\mu} = \frac{1}{\mu}\left(\sum_{k=0}^{\infty} kP_k + \sum_{k=0}^{\infty} P_k\right) = \frac{1}{\mu}(\overline{N}+1) = \frac{1}{\mu}\left(\frac{\rho}{1-\rho} + 1\right) = \frac{1}{\mu(1-\rho)}$$

b)

$$T = P_0\frac{1}{\mu} + (1 - P_0)(\frac{1}{\mu} + \lambda\frac{1}{\mu}\frac{1}{\mu} + \lambda(\lambda\frac{1}{\mu}\frac{1}{\mu})\frac{1}{\mu} + \cdots + \frac{1}{\mu}) = \quad = P_0\frac{1}{\mu} + (1 - P_0)(\frac{1}{\mu} + \frac{1}{\mu}\frac{1}{1 - \frac{\lambda}{\mu}}) =$$

$$= \frac{1}{\mu} - P_0\frac{1}{\mu(1-\rho)} + \frac{1}{\mu(1-\rho)} = \frac{1}{\mu(1-\rho)}$$

c)

Using Little's law

$$\overline{N} = \lambda T \quad \Rightarrow \quad T = \frac{\overline{N}}{\lambda} = \frac{\rho}{\lambda(1-\rho)} = \frac{1}{\mu(1-\rho)} \qquad \text{(regardless of service strategy)}$$

d)

$$N \tilde{\ } \text{Geom}(\rho) \qquad P_n = \rho^n(1-\rho)$$

$$G_N(z) = \sum_{n=0}^{\infty} P_n \cdot z^n = \sum_{n=0}^{\infty} \rho^n(1-\rho)z^n = (1-\rho)\sum_{n=0}^{\infty}(\rho z)^n = \frac{1-\rho}{1-\rho z}, \quad \rho = \frac{\lambda}{\mu}$$

$$E(N) = G_N'(1)$$

$$E(N^2) = G_N'(1) + G_N''(1)$$

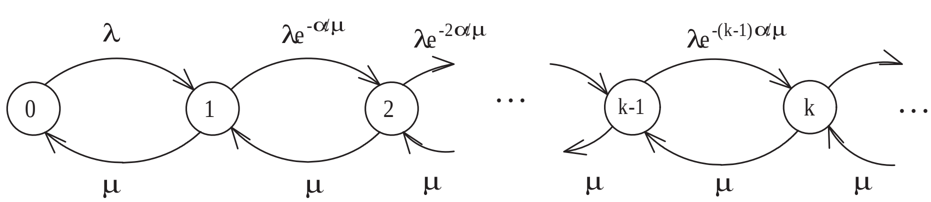
$$G_N'(z) = \frac{\rho(1-\rho)}{(1-\rho z)^2}$$

$$G_N''(z) = \frac{z\rho^2(1-\rho)}{(1-\rho z)^3}$$

$$E(N^2) = \frac{2\rho^2}{(1-\rho)^2} + \frac{\rho}{1-\rho} = \frac{\rho(1+\rho)}{(1-\rho)^2}$$

$$\text{Var}(N) = E(N^2) - E(N)^2 = \frac{\rho(1+\rho)}{(1-\rho)^2} - \left(\frac{\rho}{1-\rho}\right)^2 = \frac{\rho}{(1-\rho)^2}$$

Ex. 5.3.



$$P(\text{leave immediately}) = 1 - e^{-\alpha w}, \quad w = \frac{k}{\mu}$$

a)

$$\lambda e^{-(k-1)\frac{\lambda}{\mu}} P_{k-1} = \mu P_k \quad \Rightarrow \quad P_k = \rho e^{-(k-1)\frac{\lambda}{\mu}} P_{k-1}$$

$$P_k = \rho^k e^{-\frac{\lambda}{\mu}\sum_{i=0}^{k-1} i} P_0$$

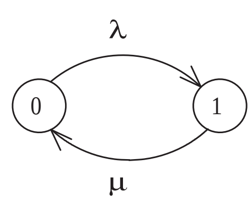
b)

If $\lambda \to \infty \quad \Rightarrow \quad$ two-state machine

$$\lambda P_0 = \mu P_1 \quad \Rightarrow \quad P_1 = \rho P_0$$

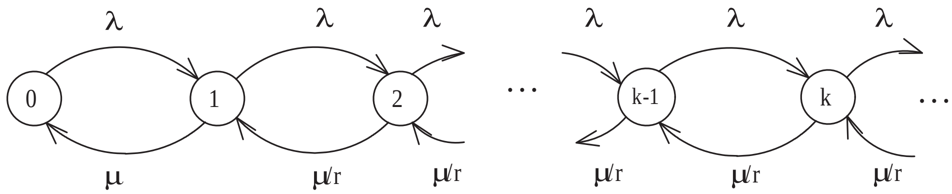
$$P_0 + P_1 = 1 \quad \Rightarrow \quad P_0 = \frac{1}{1+\rho}, \quad P_1 = \frac{\rho}{1+\rho}$$

$$\overline{N} = P_1 = \frac{\rho}{1+\rho}$$



Ex. 5.4.

a)



b)

$$P(z) = \sum_{k=0}^{\infty} P_k z^k$$

$$\lambda P_{k-1} = \frac{\mu}{r} P_k \quad \Rightarrow \quad P_k = \frac{r\lambda}{\mu} P_{k-1} = r^{k-1} \cdot \left(\frac{\lambda}{\mu}\right)^k P_0 \quad \text{for } k > 0$$

$$P(z) = P_0 + \sum_{k=1}^{\infty} r^{k-1} \cdot \left(\frac{\lambda z}{\mu}\right)^k P_0 = P_0 \left(1 + \frac{\lambda}{\mu} z \sum_{k=1}^{\infty} \left(r \frac{\lambda z}{\mu}\right)^{k-1}\right) =$$

$$= P_0 \left(1 + \frac{\frac{\lambda z}{\mu}}{1 - \frac{r\lambda z}{\mu}}\right) = P_0 \left(\frac{\lambda z(1 - \frac{1}{r}) - \frac{\mu}{r}}{\lambda z - \frac{\mu}{r}}\right)$$

$$P(1) = 1 \quad \Rightarrow \quad P_0 = \frac{\lambda - \frac{\mu}{r}}{\lambda - \frac{\lambda}{r} - \frac{\mu}{r}}$$

c)

$$\overline{N} = \lim_{z \to 1} \frac{\partial P}{\partial z}$$

$$P(z)\left(\lambda z - \frac{\mu}{r}\right) = P_0 \left(\lambda z \left(1 - \frac{1}{r}\right) - \frac{\mu}{r}\right)$$

$$\frac{\partial P}{\partial z}\left(\lambda z - \frac{\mu}{r}\right) + P(z)\lambda = P_0 \lambda \left(1 - \frac{1}{r}\right) \Big/ \lim_{z \to 1} (\text{left and right})$$

$$\overline{N}\left(\lambda - \frac{\mu}{r}\right) + \lambda = P_0 \lambda \left(1 - \frac{1}{r}\right)$$

$$\overline{N} = \frac{P_0 \lambda \left(1 - \frac{1}{r}\right) - \lambda}{\lambda - \frac{\mu}{r}}$$

Substitute $r = 1$. What do you get?

d)

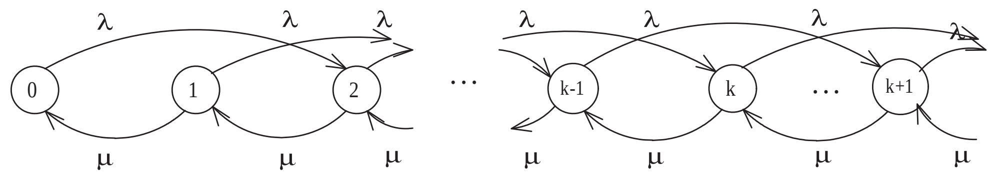
Since the arrivals are Poisson, the probabilities are equal (due to the PASTA property):

$$P_k^* = P_k = r^{k-1} \cdot \left(\frac{\lambda}{\mu}\right)^k \frac{\lambda - \frac{\mu}{r}}{\lambda - \frac{\lambda}{r} - \frac{\mu}{r}}$$

Ex. 5.5.

M/M/1 system

a)



b)

$$\lambda P_0 = \mu P_1$$

$$\mu P_k = \lambda(P_{k-1} + P_{k-2}), \qquad k \geq 1, \quad P_{-1} = 0$$

c)

$$\mu \sum_{k=1}^{\infty} P_k z^k = \lambda \sum_{k=1}^{\infty} P_{k-1} z^k + \lambda \sum_{k=2}^{\infty} P_{k-2} z^k$$

$$\mu(P(z) - P_0) = \lambda z P(z) + \lambda z^2 P(z)$$

$$P(z) = \frac{\mu P_0}{\mu - \lambda z(1+z)}$$

$$P(1) = 1 \quad \Rightarrow \quad \frac{\mu P_0}{\mu - 2\lambda} = 1 \quad \Rightarrow \quad \mu P_0 = \mu - 2\lambda$$

$$\Rightarrow P(z) = \frac{\mu - 2\lambda}{\mu - \lambda z(1+z)}$$

d)

$$P(z)[\mu - \lambda z(1+z)] = \mu - 2\lambda$$

$$\overline{N} = \lim_{z\to 1} \frac{\partial P(z)}{\partial z}$$

$$\frac{\partial P(z)}{\partial z}[\mu - \lambda z(1+z)] + P(z)[-\lambda(1+z) - \lambda z] = 0 \Big/ \lim_{z\to 1}(\text{left and right})$$

$$\Rightarrow \overline{N} = \frac{3\lambda}{\mu - 2\lambda}$$

Simple M/M/1: $\overline{N} = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}$

Ex. 5.6.

$$k\mu P_k = \lambda P_{k-1}, \qquad (k = 1, 2, \ldots)$$

$$P_k = \frac{\rho}{k} P_{k-1} = \frac{\rho}{k}\frac{\rho}{k-1} \cdot \ldots \cdot \frac{\rho}{1} P_0 = \frac{\rho^k}{k!} P_0, \qquad \rho = \frac{\lambda}{\mu}$$

$$P_k = \frac{\rho^k}{k!} P_0, \qquad (k = 1, 2, \ldots) \qquad \sum_{k=0}^{\infty} P_k = 1, \qquad P_0 e^{\rho} = 1 \quad \Rightarrow \quad P_0 = e^{-\rho}$$

$$P_k = \frac{\rho^k}{k!}e^{-\rho}, \qquad (k = 1, 2, \dots)$$

$$\overline{N} = \sum_{k=0}^{\infty} kP_k = \sum_{k=1}^{\infty} k\frac{\rho^k}{k!}e^{-\rho} = \rho e^{-\rho}\sum_{k=1}^{\infty}\frac{\rho^{k-1}}{(k-1)!} = \rho e^{-\rho}e^{\rho} = \rho$$

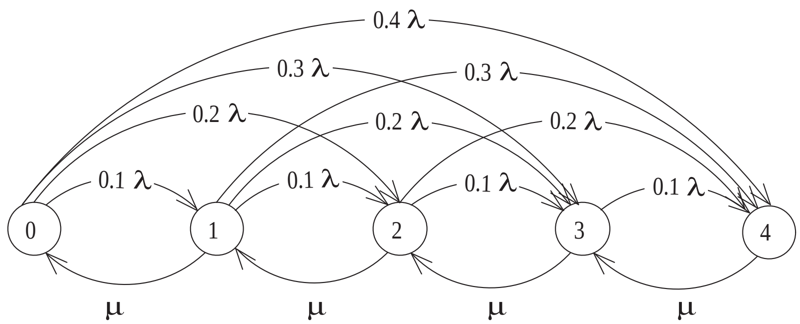
Ex. 5.7.

$$\lambda_E = 75 \text{ jobs/h}$$

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 7.5 \text{ jobs/h} = \lambda$$

$$\bar{x} = 0.5 \text{ min}, \quad \mu = 120 \text{ jobs/h}$$

a)
M/M/1/4



b)
State probabilities from global balance equations:

$$P_0 = 0.564, \quad P_1 = 0.141, \quad P_2 = 0.132, \quad P_3 = 0.105, \quad P_4 = 0.059$$

$$\overline{N}_q = \sum_{i=1}^{4}(i-1)\cdot P_i = 0.519, \qquad \overline{N} = \sum_{i=1}^{4} i\cdot P_i = 0.954$$

$$\rho = \overline{N} - \overline{N}_q, \quad \lambda_{eff} = \rho\mu, \quad \Rightarrow \quad W = \frac{\overline{N}_q}{\lambda_{eff}} = 0.59 \text{ min}$$

c)

$$P(\text{full}) = P_4$$

$$P(\text{G1 blocked}) = P_4$$

$$P(\text{G2 blocked}) = P_4 + P_3$$

$$P(\text{G3 blocked}) = P_4 + P_3 + P_2$$

$$P(\text{G4 blocked}) = 1 - P_0$$

d)

$$P(\text{blocking of customers}) = 1 - \frac{\lambda_{eff}}{\lambda} = 0.3018$$

$$P(\text{blocking of groups}) = 0.25P_1 + 0.5P_2 + 0.75P_3 + P_4 = 0.235$$

e)

$$W_3 = \frac{P_0}{P_0+P_1}W_3(\text{arriving in } P_0) + \frac{P_1}{P_0+P_1}W_3(\text{arriving in } P_1) =$$

$$= \frac{P_0}{P_0+P_1}\frac{0+0.5+1}{3} + \frac{P_1}{P_0+P_1}\frac{0.5+1+1.5}{3} = 0.6 \text{ min}$$

6. M/M/m/m loss systems

Ex. 6.1.

P(first $m-1$ servers busy)=P(first $m-1$ servers busy and m-th server idle)+P(first $m-1$ servers busy and m-th server busy). Since servers are selected in strict order, the first $m-1$ servers are an M/M/m-1/m-1 loss system. Therefore the left hand side is $E_{m-1}(\rho)$. The second term of the right hand side is $E_m(\rho)$, and thus the expected part of time that the first $m-1$ servers are busy and m-th is idle is $E_{m-1}(\rho) - E_m(\rho)$.

Ex. 6.2.

$$\rho = \frac{\lambda}{\mu} = 2$$

a)

Time blocking: $E_3(2) = 0.210526$

b)

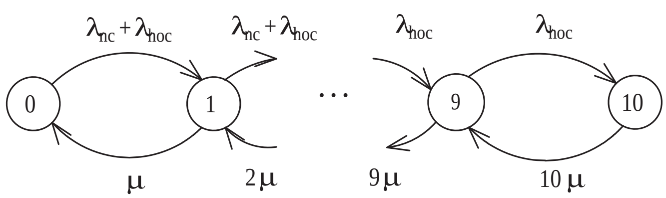
$$\lambda \cdot E_3(2) \cdot 3600 = 757.9$$

c)

Offered load: $\rho = 2$, blocked: $\rho E_3(2)$, served: $\rho(1 - E_3(2))$

Ex. 6.3.

a)



b)

M/M/10/10: λ=175 calls/hour=2.92 calls/min, $\rho = \frac{\lambda}{\mu} = \frac{2.92}{0.5} = 5.8$
From Erlang table $E_{10}(5.8) = 0.037$

$$\overline{N} = \rho(1 - E_{10}(5.8)) = 5.6$$

c)

One reserved channel:

$$P_{10} = \frac{\frac{\lambda_{hoc}}{10\mu}\frac{\lambda^9}{9!\mu^9}}{1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2!\mu} + ... + \frac{\lambda^9}{9!\mu^9} + \frac{\lambda_{hoc}}{10\mu}\frac{\lambda^9}{9!\mu^9}} = \frac{\frac{\lambda_{hoc}}{10\mu}}{\frac{1+\frac{\lambda}{\mu}+\frac{\lambda^2}{2!\mu}+...+\frac{\lambda^9}{9!\mu^9}}{\frac{\lambda^9}{9!\mu^9}} + \frac{\lambda_{hoc}}{10\mu}} = \frac{\frac{\lambda_{hoc}}{10\mu}}{\frac{1}{E_9(\rho)} + \frac{\lambda_{hoc}}{10\mu}} \approx$$

$$\approx \frac{\lambda_{hoc}}{10\mu} E_9(\rho) = \frac{0.83}{10 \cdot 0.5} \cdot E_9(5.8) = 1.1\%$$

Two reserved channels:

$$P_{10} = \frac{\lambda_{hoc}^2}{10\mu 9\mu} P_8 = \frac{0.83^2}{90 \cdot 0.5} \cdot E_8(5.8) = 0.2\%$$

Two reserved channels are needed for blocking probability of handover calls below 1%. Blocking probability for new calls is $P_8 + P_9 + P_{10} = 11\%$.

Ex. 6.4.

$$\rho_1 \quad \rightarrow \quad \overbrace{\text{O} \cdots\cdots\cdots \text{O}}^{m_1} \qquad \rho_1 = 3$$

$$\rho_2 \quad \rightarrow \quad \overbrace{\text{O} \cdots\cdots\cdots \text{O}}^{m_2} \qquad \rho_2 = 6$$

$$\rho_1 + \rho_2 \quad \rightarrow \quad \overbrace{\text{O} \cdots\cdots\cdots \text{O}}^{m_3} \qquad \rho_3 = \rho_1 + \rho_2 = 9$$

a)

$$E_m(\rho) = \frac{\frac{\rho^m}{m!}}{\sum_{i=0}^{m} \frac{\rho^i}{i!}}$$

$$P(\text{blocking}) = E_m(\rho) < 0.001$$

1) $m_1 = 10 \quad E_{10}(3) = 0.000810$

2) $m_2 = 15 \quad E_{15}(6) = 0.000892$

3) $m_3 = 20 \quad E_{20}(9) = 0.000617$

b)

In an M/M/m/m system:

$$P(k) = \frac{\frac{\rho^k}{k!}}{\sum_{i=0}^{m} \frac{\rho^i}{i!}} = E_m(\rho) \cdot \frac{\frac{\rho^k}{k!}}{\frac{\rho^m}{m!}}$$

Hence:

$$\overline{N} = \sum_{k=1}^{m} k \cdot P(k) = E_m(\rho) \cdot \frac{m!}{\rho^m} \cdot \sum_{k=1}^{m} k \cdot \frac{\rho^k}{k!} = E_m(\rho) \cdot \frac{m!}{\rho^m} \cdot \rho \cdot \sum_{k=1}^{m} \frac{\rho^{k-1}}{(k-1)!} = E_m(\rho) \cdot \frac{m!}{\rho^m} \cdot \rho \cdot \sum_{k=0}^{m-1} \frac{\rho^k}{k!} =$$

$$= E_m(\rho) \cdot \frac{m!}{\rho^m} \cdot \rho \cdot \frac{\frac{\rho^{m-1}}{(m-1)!}}{E_{m-1}(\rho)} = \frac{E_m(\rho)}{E_{m-1}(\rho)} \cdot \frac{m!}{\rho^m} \cdot \rho \cdot \frac{\rho^{m-1}}{(m-1)!} = m \cdot \frac{E_m(\rho)}{E_{m-1}(\rho)} = m \frac{\frac{\frac{\rho^m}{m!}}{1+\rho+\ldots+\frac{\rho^m}{m!}}}{\frac{\frac{\rho^{m-1}}{(m-1)!}}{1+\rho+\ldots+\frac{\rho^{m-1}}{(m-1)!}}} =$$

$$= \rho(1 - E_m(\rho))$$

1) $m_1 = 10 \quad \overline{N} \approx 3$ 2) $m_2 = 15 \quad \overline{N} \approx 6$ 3) $m_3 = 20 \quad \overline{N} \approx 9$

c)

$$\text{Average utilization} = \frac{\overline{N}_s}{m} = \frac{\rho(1 - E_m(\rho))}{m}$$

1) 0.3 2) 0.4 3) 0.45

d)

$$P(\text{max. two servers idle}) = P_m + P_{m-1} + P_{m-2}$$

$$P_m = E_m(\rho) \qquad P_{m-1} = \frac{m\mu}{\lambda} P_m \qquad P_{m-2} = \frac{(m-1)\mu}{\lambda} P_{m-1}$$

1) 0.01161 2) 0.008325 3) 0.004883

Ex. 6.5.

M/M/m/m loss system, Erlang system

$$P_k = \frac{\frac{\rho^k}{k!}}{\sum_{i=0}^{m} \frac{\rho^i}{i!}}, \qquad k = 0, \ldots, m, \qquad \rho = \frac{\lambda}{\mu}$$

a)

$$\text{Blocking probability} = P_m = \frac{\frac{\rho^m}{m!}}{\sum_{i=0}^{m} \frac{\rho^i}{i!}} = E_m(\rho)$$

$$\rho = \frac{180 \cdot 110}{3600} = 5.5 \text{ Erlang}$$

$$P_{10} = E_{10}(5.5) = \text{(Erlang tables)} = 0.02926$$

b)

$R =$ number of attempted calls rejected per hour

$$R = \lambda \cdot P_{blocking} = \lambda \cdot P_{10} = 5.27 \text{ calls/hour}$$

c)

$$\text{Traffic load per server} = \frac{\rho}{m} = \frac{5.5}{10} = 0.55 \text{ Erlang}$$

d)

$$P_{blocking} = 0.02 = P_{10} = E_{10}(\rho)$$

$\Rightarrow$ Erlang tables $\rho = 5 = \rho_{max}$

$$\rho_{max} = \frac{\lambda_{max}}{\mu} \quad \Rightarrow \quad \lambda_{max} = 5 \cdot \mu = 163 \text{ calls/hour}$$

Ex. 6.6.

a)

Let $X(t)$ be the number of machines up at time t. It is easily seen that $X(t)$ is a birth and death process with birth and death rates given by $\lambda_k = \beta$ for $k = 0, 1, \ldots, K-1$, $\lambda_k = 0$ for $k \geq K$ and $\mu_k = k \cdot \alpha$ for $k = 1, 2, \ldots, K$, respectively. We notice that $X(t)$ has the same behavior as the queue-length process of an M/M/K/K queue! Hence:

$$P(i) = \frac{\left(\frac{\beta}{\alpha}\right)^i}{i! \cdot C(K, \frac{\beta}{\alpha})}$$

for $i = 0, 1, \ldots, K$, where:

$$C(K, a) = \sum_{i=0}^{K} \frac{a^i}{i!}$$

b)

The overall failure rate λ_b is given by:

$$\lambda_b = \sum_{i=1}^{K} (i \cdot \alpha) \cdot P(i) = \frac{\alpha}{C(K, \frac{\beta}{\alpha})} \cdot \sum_{i=1}^{K} \frac{i \cdot \left(\frac{\beta}{\alpha}\right)^i}{i!}$$

c)

Utilization ρ is given by

$$\rho = \frac{\lambda_b}{\beta} = \frac{\alpha}{\beta} \cdot \frac{1}{C(K, \frac{\beta}{\alpha})} \cdot \sum_{i=1}^{K} \frac{i \cdot \left(\frac{\beta}{\alpha}\right)^i}{i!} = \frac{1}{C(K, \frac{\beta}{\alpha})} \cdot \sum_{i=0}^{K-1} \frac{(i+1) \cdot \left(\frac{\beta}{\alpha}\right)^i}{(i+1)!} = \frac{C(K-1, \frac{\beta}{\alpha})}{C(K, \frac{\beta}{\alpha})}$$

d)

$$P(0) = \frac{1}{C(K, \frac{\beta}{\alpha})}$$

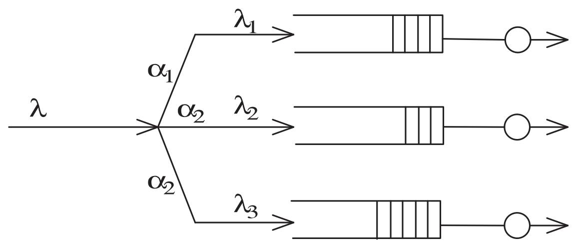
e)
Number of required computers N is calculated from the condition:

$$\sum_{i=1}^{N} i \cdot P(i) \geq K \qquad \Rightarrow \qquad \frac{\beta}{\alpha} \cdot \frac{C(N-1, \frac{\beta}{\alpha})}{C(N, \frac{\beta}{\alpha})} \geq K$$

7. M/M/m queuing systems

Ex. 7.1.

a)



$$\alpha_i = \text{prob. of choosing counter } i = \frac{1}{3}, \qquad (i = 1, 2, \cdots)$$

From 3.2:

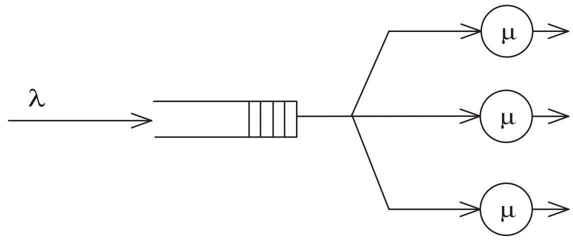
$$\lambda_1 = \lambda\alpha_1, \quad \lambda_2 = \lambda\alpha_2, \quad \lambda_3 = \lambda\alpha_3, \quad \lambda_i = \frac{\lambda}{3} \qquad (i = 1, 2, \cdots)$$

Each queue is M/M/1 system:

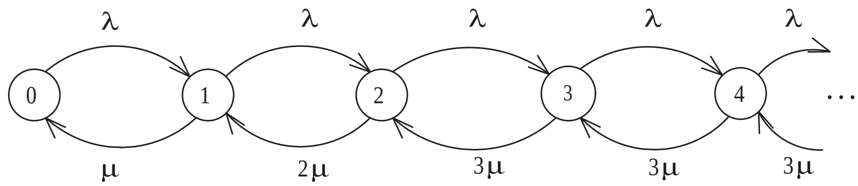
$$\overline{T} = \frac{\overline{N}}{\lambda_1}, \quad \overline{N} = \frac{\rho}{1-\rho}, \quad \rho = \frac{\lambda_1}{\mu} = \lambda_1\bar{x}, \quad \bar{x} = \frac{1}{\mu} = 45s$$

$$\overline{T} = \frac{\bar{x}}{1 - \lambda_1\bar{x}} = 77.14s \quad \text{(make sure all units match!)}$$

b)



M/M/3 system: $\overline{T} = \frac{\overline{N}}{\lambda}, \quad \overline{x} = \frac{1}{\mu} = 45sec$



$$P_k = \begin{cases} \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k P_0 & k \le m \\ \frac{1}{m! m^{k-m}}\left(\frac{\lambda}{\mu}\right)^k P_0 & k > m \end{cases}$$

$$\sum_{k=0}^{\infty} P_k = 1 \quad \Rightarrow \quad P_0 = \left[\sum_{k=0}^{m-1} \frac{\rho^k}{k!} + \frac{\rho^m}{m!}\frac{m}{m-\rho}\right]^{-1} = \left[\sum_{k=0}^{m-1} \frac{\rho^k}{k!} + P_q\right]^{-1}, \qquad \rho = \frac{\lambda_1}{\mu}$$

$$P_q = D_m(\rho) = \frac{P_0 \rho^m}{m!}\frac{m}{m-\rho} \quad \text{(Erlang-C formula)}$$

For M/M/3 one can compute P_q easily, but for larger m computation is more difficult, so we use Erlang tables in general:

$$D_m(\rho) = \frac{m E_m(\rho)}{m - \rho(1 - E_m(\rho))}, \quad E_m(\rho) = \frac{\frac{\rho^m}{m!}}{\sum_{k=0}^{m} \frac{\rho^k}{k!}}$$

$$\overline{N} = \rho + D_m(\rho)\frac{\rho}{m-\rho}, \qquad \Rightarrow \qquad \overline{T} = \frac{\overline{N}}{\lambda} = \frac{1}{\mu} + \frac{D_m(\rho)}{\mu(m-\rho)}$$

From table:

$$E_3(1.38) = 0.119180 \quad \Rightarrow \quad D_3(1.38) = 0.201 \quad \Rightarrow \quad \overline{T} = 56.16sec$$

Ex. 7.2.

a)

Little: $\overline{N} = \lambda \overline{T}$ $\quad \overline{T} = 2.5$ sec

N : M/M/1: $\overline{N} = \sum_{k=0}^{\infty} k P_k = \frac{\rho}{1-\rho}, \qquad \rho = \frac{\lambda}{\mu}, \qquad \frac{1}{\mu} = 0.5$

T : $\overline{T} = \frac{1}{\mu - \lambda}$ $\quad \overline{T} = 2.5$ sec $\quad \lambda = 1.6$

b)

$$\lambda' = 0.1\lambda + \lambda = 1.76 \qquad \overline{T}' = 4.17 \text{ s} \qquad \text{Increase } 67\%$$

c)

$$\overline{W} = \overline{T} - \frac{1}{\mu} = \frac{\rho}{\mu - \lambda} = 2 \text{ s}, \quad N_s = \rho = 0.8, \quad P_0 = 1 - \rho = 0.2$$

d)

$$\mu = 1 \qquad \lambda = 1.549$$

$$\overline{T} = \frac{1}{\mu} + \overline{W} = \frac{1}{\mu} + \frac{D_2(\rho)}{\mu(2-\rho)}$$

$$D_2(\rho) = \frac{2E_2(\rho)}{2 - \rho(1 - E_2(\rho))}, \quad E_2(\rho) = \frac{\frac{\rho^2}{2}}{1 + \rho + \frac{\rho^2}{2}}$$

$$\Rightarrow \overline{T} = \frac{4\mu}{4\mu^2 - \lambda^2} = 2.5 \text{ sec}$$

e)

$$\lambda' = 0.1\lambda + \lambda = 1.704 \qquad \overline{T} = 3.65 \text{ s} \qquad \text{Increase } 46\%$$

f)

$$\overline{W} = \overline{T} - \frac{1}{\mu} = 1.5 \text{ s}$$

$$\overline{N}_s = \rho = \frac{\lambda}{\mu} = 1.549, \quad \text{Utilization} = \frac{\overline{N}_s}{m} = \frac{\rho}{2} = 77.45\%$$

$$P(\text{wait}) = P_q = D_2(\rho) = \frac{2 \cdot E_2(\rho)}{2 - \rho(1 - E_2(\rho))} = \frac{2 \cdot 0.32}{2 - 1.549(1 - 0.32))} = 67.65\%$$

Ex. 7.3.

The waiting time for our customer is $W = X_1 + X_2 + X_3$, where X_1 is the time elapsed since the arrival of the customer until the first service event (departure), X_2 is the time from the first departure until the second departure and X_3 is the time from the second until the third departure. At the third departure, our customer starts to be served and its waiting time ends. Because of the memoryless property of the exponential distribution and the independence of all service times, we conclude that inter-departure times X_1, X_2, and X_3 are also independent.
The inter-departure time distribution is:

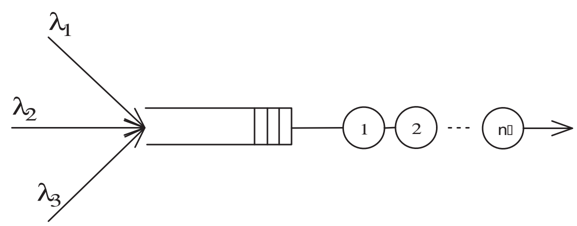
$$P(X_i > t)\text{=P(min. of 3 independent exp. dist. variables} > t) = e^{-\mu t} \cdot e^{-\mu t} \cdot e^{-\mu t} = e^{-3\mu t},$$

where $\frac{1}{\mu} = 1.25 \quad (i = 1, 2, 3)$. Therefore, inter-departure times are exponentially distributed and departure process is Poisson with rate 3μ :

$$P(W > t) = P(\text{max. two events in a Poisson process with intensity } 3\mu \text{ in interval } t) =$$
$$= e^{-3\mu t} + 3\mu t e^{-3\mu t} + \frac{(3\mu t)^2}{2} e^{-3\mu t}$$

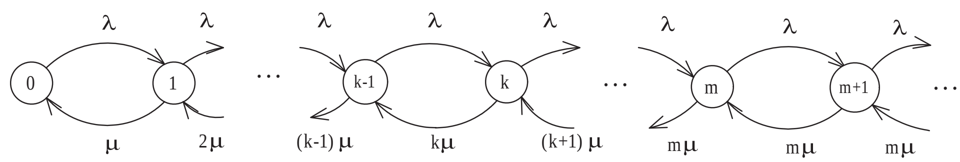
With $\frac{1}{\mu} = 1.25$ and $t = 1.25$, we obtain $P(W > 1.25) = e^{-3} + 3e^{-3} + 4.5e^{-3} = 8.5e^{-3} \approx 0.42$.

Ex. 7.4.



$$\lambda = \lambda_1 + \lambda_2 + \lambda_3, \qquad \rho = \frac{\lambda}{\mu}$$

a)



b)

$$\lambda P_{k-1} = k\mu P_k, \qquad k \le m$$

$$\lambda P_{k-1} = m\mu P_k, \qquad k > m$$

$$P_k = \frac{\lambda}{k\mu} P_k = \ldots = \frac{\rho^k}{k!} P_0, \qquad k \le m$$

$$P_k = \frac{\lambda}{m\mu} P_k = \ldots = \left(\frac{\lambda}{m\mu}\right)^{k-m} P_m = \frac{\rho^k}{m!}\left(\frac{1}{m}\right)^{k-m} P_0, \qquad k > m$$

$$P_k = \begin{cases} \frac{\rho^k}{k!} P_0 & k \le m \\ \frac{\rho^k}{m!}\left(\frac{1}{m}\right)^{k-m} P_0 & k > m \end{cases}$$

$$\sum P_k = 1 \quad \Rightarrow \quad P_0 = \left[\sum_{k=0}^{m-1} \frac{\rho^k}{k!} + \sum_{k=m}^{\infty} \frac{\rho^k}{m!}\left(\frac{1}{m}\right)^{k-m}\right]^{-1} =$$

$$= \left[\sum_{k=0}^{m-1} \frac{\rho^k}{k!} + \frac{\rho^m}{m!} \sum_{k=m}^{\infty} \left(\frac{\rho}{m}\right)^{k-m}\right]^{-1} = \left[\sum_{k=0}^{m-1} \frac{\rho^k}{k!} + \frac{\rho^m}{m!} \frac{m}{m-\rho}\right]^{-1}$$

c)

$$\overline{N} = \sum_{k=0}^{m} k P_k = P_0 \sum_{k=0}^{m} k \frac{\rho^k}{k!} + P_0 \sum_{k=m+1}^{\infty} k \frac{\rho^m}{m!}\left(\frac{\rho}{m}\right)^{k-m} = P_0 \rho \sum_{k=0}^{m-1} \frac{\rho^k}{k!} + P_0 \sum_{k=m+1}^{\infty} k \frac{\rho^m}{m!}\left(\frac{\rho}{m}\right)^{k-m}$$

$$P_0 \sum_{k=0}^{m-1} \frac{\rho^k}{k!} + \sum_{k=m}^{\infty} P_k = 1, \qquad \sum_{k=m}^{\infty} P_k = D_m(\rho) \quad \Rightarrow \quad P_0 \rho \sum_{k=0}^{m-1} \frac{\rho^k}{k!} = \rho - \rho D_m(\rho)$$

$$\overline{N} = \rho - \rho D_m(\rho) + P_0 \frac{\rho^m}{m!} \sum_{k=m+1}^{\infty} k \left(\frac{\rho}{m}\right)^{k-m}$$

$$P_0 \frac{\rho^m}{m!} \sum_{k=m+1}^{\infty} k \left(\frac{\rho}{m}\right)^{k-m} = P_0 \frac{\rho^m}{m!} \left[\sum_{k=m+1}^{\infty} (k-m) \left(\frac{\rho}{m}\right)^{k-m} + m \sum_{k=m+1}^{\infty} \left(\frac{\rho}{m}\right)^{k-m}\right] =$$

$$= P_0 \frac{\rho^m}{m!}\left[\sum_{k=1}^{\infty} k\left(\frac{\rho}{m}\right)^k + m\sum_{k=1}^{\infty}\left(\frac{\rho}{m}\right)^k\right] = P_0\frac{\rho^m}{m!}\left[\frac{\frac{\rho}{m}}{\left(1-\frac{\rho}{m}\right)^2} + \frac{\rho}{1-\frac{\rho}{m}}\right]$$

$$\Rightarrow \overline{N} = \rho - \rho D_m(\rho) + P_0\frac{\rho^m}{m!}\left[\frac{\rho m}{(m-\rho)^2} + \frac{m\rho}{m-\rho}\right]$$

$$D_m(\rho) = P_0\frac{\rho^m}{m!}\frac{m}{m-\rho}$$

$$\Rightarrow \overline{N} = \rho - \rho D_m(\rho) + \frac{\rho}{m-\rho}D_m(\rho) + \rho D_m(\rho) = \rho + \frac{\rho}{m-\rho}D_m(\rho)$$

$$D_m(\rho) = \frac{mE_m(\rho)}{m-\rho(1-E_m(\rho))}$$

$$\lambda_1 = 3\frac{1}{\text{min}}, \quad \lambda_2 = 2\frac{1}{\text{min}}, \quad \lambda_3 = 1\frac{1}{\text{min}}, \quad \mu = 0.1\frac{1}{\text{sec}}$$

$$m = 2 \qquad \rho = \frac{\lambda}{\mu} = \frac{\lambda_1+\lambda_2+\lambda_3}{\mu} = 1$$

$$\Rightarrow \quad E_2(1) = 0.2 \quad \Rightarrow \quad D_2(1) = \frac{1}{3} \quad \Rightarrow \quad \overline{N} = 1 + \frac{1}{3} = \frac{4}{3}$$

d)

$$\overline{N}_1 = \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}\overline{N} = \frac{2}{3}$$

e)
Little's law

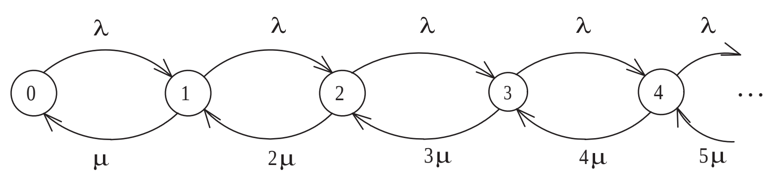
$$\overline{N} = \lambda T = \lambda\bar{x} + \lambda\overline{W}$$

$$\overline{W} = \frac{\overline{N}-\rho}{\lambda} = \frac{1}{m(m-\rho)}D_m(\rho) = \frac{\frac{4}{3}-1}{6} = \frac{1}{18}\text{ min}$$

f) Common queue ⇒ waiting time does not depend on job type

$$\overline{W}_1 = \overline{W} = \frac{1}{18}\text{ min}$$

Ex. 7.5.



$$P_k = \frac{\rho^k}{k!}P_0, \qquad \sum P_k = 1 \quad \Rightarrow \quad P_0 = e^{-\rho}$$

$$P_k = \frac{\rho^k}{k!}e^{-\rho}$$

a)
Served traffic $= \rho$.

b)
Mean waiting time $\overline{W} = 0$ (mean time in the system $= \frac{1}{\mu}$).

c)

$$P_k = \frac{\rho^k}{k!}e^{-\rho}$$

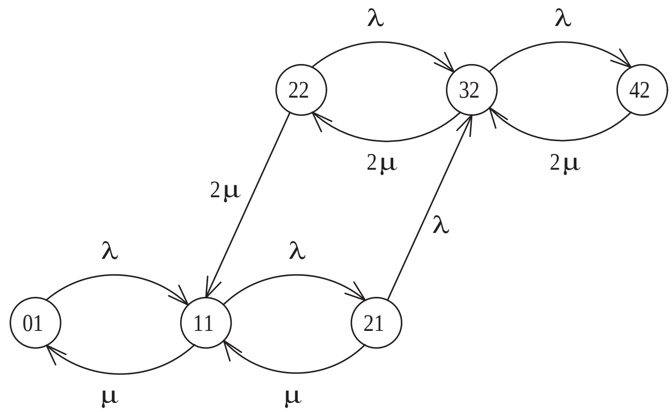
d)

$$\overline{N} = \sum_{k=1}^{\infty} kP_k = \sum_{k=1}^{\infty} k\frac{\rho^k}{k!}e^{-\rho} = \sum_{k=1}^{\infty}\rho\frac{\rho^{k-1}}{(k-1)!}e^{-\rho} = \rho e^{-\rho}\sum_{k=0}^{\infty}\frac{\rho^k}{k!} = \rho e^{-\rho}e^{\rho} = \rho$$

Ex. 7.6.

a)

M/M/2
$\lambda = 3$
$\mu = 3$
$\rho = \frac{\lambda}{\mu} = 1$



b)

$$P_{01} \cdot \lambda = P_{11} \cdot \mu \qquad \Rightarrow \qquad P_{11} = P_{01} \cdot \frac{\lambda}{\mu}$$

$$P_{11}(\lambda + \mu) = P_{01} \cdot \lambda + P_{21} \cdot \mu + P_{22} \cdot 2\mu$$

$$P_{21}(\lambda + \mu) = P_{11} \cdot \lambda \qquad \Rightarrow \qquad P_{21} = P_{11} \cdot \frac{\lambda}{\lambda + \mu} = P_{01} \cdot \frac{\lambda}{\mu} \cdot \frac{\lambda}{\lambda + \mu}$$

$$P_{22}(2\lambda + \mu) = P_{32} \cdot 2\mu \qquad \Rightarrow \qquad P_{32} = P_{22} \cdot \frac{2\mu + \lambda}{2\mu}$$

$$P_{22} \cdot 2\mu = P_{21} \cdot \lambda \qquad \Rightarrow \qquad P_{22} = P_{21} \cdot \frac{\lambda}{2\mu} = P_{01} \cdot \frac{\lambda}{\mu} \cdot \frac{\lambda}{2\mu} \cdot \frac{\lambda}{\lambda + \mu}$$

$$P_{i2} \cdot \lambda = P_{i+1,2} \cdot 2\mu \qquad \Rightarrow \qquad P_{i2} = P_{32} \cdot \frac{1}{2^{i-3}} \cdot \rho^{i-3}, \quad (i \geq 3)$$

$$\sum_i \sum_j P_{ij} = 1, \qquad \sum_{i=3}^{\infty} P_{i2} = \sum_{i=0}^{\infty} \left(\frac{\rho}{2}\right)^i \cdot P_{32} = \frac{1}{1 - \frac{\rho}{2}} \cdot P_{32}$$

$$\Rightarrow \qquad P_{01} = \frac{2}{7}, \quad P_{11} = \frac{2}{7}, \quad P_{21} = \frac{1}{7}, \quad P_{22} = \frac{1}{14}, \quad P_{32} = \frac{3}{28}, \quad P_{i2} = P_{32} \frac{1}{2^{i-3}} \rho^{i-3} \quad (i \geq 3)$$

c)

$$N_q = P_{21} \cdot 1 + P_{32} \cdot 1 + P_{42} \cdot 2 + \ldots = P_{21} + P_{32} \cdot \sum_{i=0}^{\infty} (i+1) \left(\frac{\rho}{2}\right)^i =$$

$$= P_{21} + P_{32} \left[\sum_{i=0}^{\infty} \left(\frac{\rho}{2}\right)^i + \sum_{i=0}^{\infty} i \cdot \left(\frac{\rho}{2}\right)^i \right] = \frac{1}{7} + \frac{3}{14} + \frac{6}{28} = \frac{4}{7}$$

$$N_s = P_{11} \cdot 1 + P_{21} \cdot 1 + P_{22} \cdot 2 + 2 \cdot P_{32} \cdot \sum_{i=0}^{\infty} \left(\frac{\rho}{2}\right)^i = \frac{2}{7} + \frac{1}{7} + 2 \cdot \frac{1}{14} + 2 \cdot \frac{3}{14} = 1$$

d)
Erlang (i-2) distributed with mean $(i-2)\frac{1}{2\mu}$.

e)
After the next arrival or service finishes, the waiting time is the minimum of $\exp(\lambda)$ and $\exp(\mu)$, and thus it is $\exp(\lambda + \mu)$, i.e. exponentially distributed with parameter $\lambda + \mu = 6$.

8. M/M/m with limited system capacity and finite customer population (M/M/n/S and M/M/m/*/C)

Ex. 8.1.

For M/M/1/K:

$$P_k = \frac{\rho^k(1-\rho)}{1-\rho^{K+1}}, \qquad \rho = \frac{\lambda}{\mu}, \qquad k = (0, 1, \ldots, K)$$

a)

$$P_0 = \frac{1-\rho}{1-\rho^{K+1}}$$

b)

$$1 - P_0 - P_K = \frac{\rho - \rho^K}{1-\rho^{K+1}}$$

c)

$$P_K = \frac{\rho^K(1-\rho)}{1-\rho^{K+1}}$$

Ex. 8.2.

$$\lambda P_0 = \mu P_1; \qquad \lambda P_1 = 2\mu P_2; \qquad \lambda P_2 = 2\mu P_3$$

$$\sum_{k=0}^{3} P_k = 1, \qquad \rho = \frac{\lambda}{\mu}$$

$$P_1 = \rho P_0; \qquad P_2 = \frac{\rho^2}{2} P_0; \qquad P_3 = \frac{\rho^3}{4} P_0; \qquad P_0 = \frac{4}{4 + 4\rho + 2\rho^2 + \rho^3}$$

$$\lambda = 2 \text{ arrivals/min}; \qquad \frac{1}{\mu} = \frac{45}{60} \text{ min}; \qquad \rho = \frac{3}{2}; \qquad P_0 = \frac{32}{143}$$

Probability that a customer has to wait $= P_2 = \frac{36}{143} \approx 0.252$

Probability that a customer leaves the system $= P_3 = \frac{27}{143} \approx 0.189$

Probability that a customer has to wait longer than t sec:

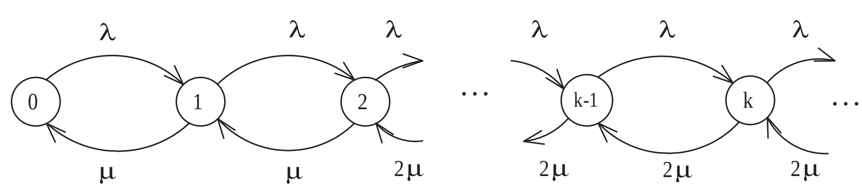
$$P(W > t) = \sum_{k=0}^{3} P(W > t|k) P_k = P(W > t|2) P_2 = P(min(X_1, X_2) > t) P_2 =$$

$$= P(X_1 > t) P(X_2 > t) P_2 = e^{-2\mu t} P_2 = \frac{2\rho^2 e^{-2\mu t}}{4 + 4\rho + 2\rho^2 + \rho^3}, \qquad (t \geq 0)$$

Ex. 8.3.

a)

M/M/m



b)

$$\lambda P_{k-1} = \mu P_k, \qquad k = 1, 2 \qquad\qquad \lambda P_{k-1} = 2\mu P_k, \qquad k = 3, \ldots$$

c)

$$P_k = \left(\frac{\lambda}{\mu}\right)^k P_0, \qquad k = 0,1$$

$$\sum P_k = 1$$

$$P_k = \left(\frac{\lambda}{2\mu}\right)^{k-2}\left(\frac{\lambda}{\mu}\right)^2 P_0, \qquad k = 2,3,\ldots$$

$$\Rightarrow \qquad P_0 + \frac{\lambda}{\mu}P_0 + P_0\left(\frac{\lambda}{\mu}\right)^2 \sum_{k=2}^{\infty}\left(\frac{\lambda}{2\mu}\right)^{k-2} = 1 \qquad \Rightarrow \qquad P_0 = \frac{1}{1+\frac{\lambda}{\mu}+\frac{\left(\frac{\lambda}{\mu}\right)^2}{1-\frac{\lambda}{2\mu}}}$$

$$\Rightarrow \qquad P_0 = \frac{1}{1+\rho+\frac{\rho^2}{1-\frac{\rho}{2}}} = \frac{2-\rho}{2+\rho+\rho^2} \qquad P_k = \begin{cases} \rho^k \cdot \frac{2-\rho}{2+\rho+\rho^2} & k = 0,1 \\ \rho^k \cdot \frac{1}{2^{k-2}}\frac{2-\rho}{2+\rho+\rho^2} & k = 2,3,\ldots \end{cases}$$

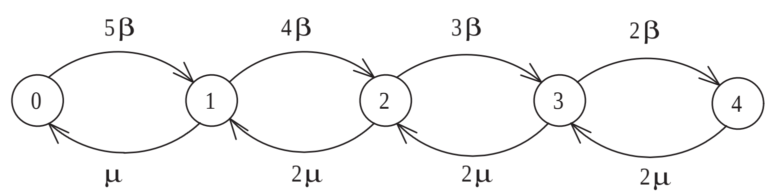
d)

$$\overline{N}_S = P_1 + P_2 + 2\sum_{k=3}^{\infty} P_k = P_1 + P_2 + 2(1 - P_0 - P_1 - P_2) = 2(1-P_0) - (P_1+P_2) =$$

$$= 2(1-P_0) - (\rho P_0 + \rho^2 P_0) = 2 - P_0(2+\rho+\rho^2) = 2 - (2-\rho) = \rho$$

Ex. 8.4.

a)



b)

$$P_1 = \frac{5\beta}{\mu}P_0 = 5P_0 \qquad P_2 = \frac{4\beta}{2\mu}P_1 = 10P_0 \qquad P_3 = \frac{3\beta}{2\mu}P_2 = 15P_0$$

$$P_4 = \frac{2\beta}{2\mu}P_3 = 15P_0 \qquad \sum_{k=0}^{4} P_k = 1$$

$$\Rightarrow \qquad P_0 = \frac{1}{46}, \qquad P_1 = \frac{5}{46}, \qquad P_2 = \frac{10}{46}, \qquad P_3 = \frac{15}{46}, \qquad P_4 = \frac{15}{46}$$

Average number of customers in the queue

$$\overline{N}_q = 0\cdot P_0 + 0\cdot P_1 + 0\cdot P_2 + 1\cdot P_3 + 2\cdot P_4 = \frac{45}{46}$$

$$\lambda_{eff} = 5\beta P_0 + 4\beta P_1 + 3\beta P_2 + 2\beta P_3 = \frac{85}{46}$$

$$W = \frac{\overline{N}_q}{\lambda_{eff}} = \frac{45}{85} \approx 0.53$$

c)

Time blocking probability $= P_4 = \frac{15}{46} \approx 0.326$

d)

Call blocking probability $= P_4^* = \frac{\beta P_4}{5\beta P_0 + 4\beta P_1 + 3\beta P_2 + 2\beta P_3 + \beta P_4} \approx 0.15.$

The time blocking and the call blocking probability are not equal, since the arrival process is not a homogeneous Poisson process.

Ex. 8.5.

a)

A blocking system with finite number of users: Engset case (not Erlang!)

$$\text{Call blocking:} \qquad r_m = \frac{\binom{M-1}{m}\alpha^m}{\sum_{i=0}^{m}\binom{M-1}{i}\alpha^i}$$

$$\beta = \frac{1 \text{ call}}{75 \text{ minutes}} = \frac{4}{5} \text{ calls/hour}$$

$$\mu = \frac{1 \text{ call}}{15 \text{ minutes}} = 4 \text{ calls/hour}$$

$$M = 8 \qquad r_m = 0.02 \qquad \alpha = \frac{\beta}{\mu} \quad \Rightarrow \quad \alpha = \frac{1}{5}$$

$$m = 3: \qquad r_3 = 0.080 > 0.02$$

$$m = 4: \qquad r_4 = 0.016 < 0.02 \qquad \text{Four links are enough!}$$

b)

i) Erlang case ($M >> m$)

$$E_m(\rho) = 0.02 \qquad \rho = \frac{80 \text{ calls/hour}}{4 \text{ calls/hour}} = 20$$

$$E_m(20) \leq 0.02 \quad \Rightarrow \quad m = 28 \text{ links} \quad (<< M = 300!)$$

ii)

$$P(W \leq y) = 1 - D_m(\rho)e^{-\mu(m-\rho)y}$$

$$\overline{W} = \int_0^\infty [1 - P(W \leq y)]\mathrm{d}y = D_m(\rho)\int_0^\infty e^{-\mu(m-\rho)y}\mathrm{d}y = D_m(\rho)\left[-\frac{e^{-\mu(m-\rho)y}}{\mu(m-\rho)}\right]_0^\infty = \frac{D_m(\rho)}{\mu(m-\rho)}$$

$$D_m(\rho) = \frac{mE_m(\rho)}{m - \rho(1 - E_m(\rho))}$$

$$E_{23}(20) = 0.019 \quad \Rightarrow \quad D_m(\rho) = \frac{28 \cdot 0.019}{28 - 20(1 - 0.019)} = 0.063$$

$$\overline{W} = \frac{0.063}{4(28 - 20)} \approx 7 \text{ sec}$$

Ex. 8.6.

M/M/2/5/8 system
Let us denote $\rho = \frac{\alpha}{\beta} = \frac{0.5}{1} = 0.5$

a)

State probabilities:

$$P_1 = 8\rho P_0 = 4P_0 = \frac{32}{390}$$

$$P_2 = \frac{7}{2} \cdot 8\rho^2 P_0 = 28\rho^2 P_0 = \frac{56}{390}$$

$$P_3 = 3 \cdot \frac{7}{2} \cdot 8\rho^3 P_0 = 84\rho^3 P_0 = \frac{84}{390}$$

$$P_4 = \frac{5}{2} \cdot 3 \cdot \frac{7}{2} \cdot 8\rho^4 P_0 = 210\rho^4 P_0 = \frac{105}{390}$$

$$P_5 = 2 \cdot \frac{5}{2} \cdot 3 \cdot \frac{7}{2} \cdot 8\rho^5 P_0 = 420\rho^5 P_0 = \frac{105}{390}$$

$$P_0 = \frac{1}{1 + 4 + 7 + \frac{21}{2} + \frac{105}{8} + \frac{105}{8}} = \frac{8}{390}$$

b)

Mean waiting time of the student:

$$W = \frac{N_q}{\lambda_{eff}}$$

$$N_q = 0 \cdot P_0 + 0 \cdot P_1 + 0 \cdot P_2 + 1 \cdot P_3 + 2 \cdot P_4 + 3 \cdot P_5 = \frac{609}{390}$$

$$W = \frac{609}{732} = 0.83 \text{ h}$$

$$\lambda_{eff} = 8\alpha P_0 + 7\alpha P_1 + 6\alpha P_2 + 5\alpha P_3 + 4\alpha P_4 = \frac{732}{390}$$

c)
Time blocking probability:

$$P_5 = \frac{105}{390} = 0.27$$

d)
Call blocking probability:

$$P_C = \frac{3\alpha P_5}{8\alpha P_0 + 7\alpha P_1 + 6\alpha P_2 + 5\alpha P_3 + 4\alpha P_4 + 3\alpha P_5} = 0.177$$

e)
The service rate seen by a customer in the queue is $2\beta = 2$.

P(W > 2h) = P(W > 2h | arrives in state 2)·P(arrives in state 2) +
P(W > 2h | arrives in state 3)·P(arrives in state 3) +
P(W > 2h | arrives in state 4)·P(arrives in state 4)

P(W > 2h | arrives in state 2) = $\frac{(2\beta t)^0}{0!} e^{-2\beta t} = e^{-4} = 0.0183$

P(W > 2h | arrives in state 3) = $(\frac{(2\beta t)^0}{0!} + \frac{(2\beta t)^1}{1!}) \cdot e^{-2\beta t} = (1 + 4) \cdot e^{-4} = 0.0916$

P(W > 2h | arrives in state 4) = $(\frac{(2\beta t)^0}{0!} + \frac{(2\beta t)^1}{1!} + \frac{(2\beta t)^2}{2!}) \cdot e^{-2\beta t} = (1 + 4 + 8) \cdot e^{-4} = 0.2381$

P(arrives in state 2) = $\frac{6\alpha P_2}{8\alpha P_0 + 7\alpha P_1 + 6\alpha P_2 + 5\alpha P_3 + 4\alpha P_4 + 3\alpha P_5} = 0.188$

P(arrives in state 3) = $\frac{5\alpha P_3}{8\alpha P_0 + 7\alpha P_1 + 6\alpha P_2 + 5\alpha P_3 + 4\alpha P_4 + 3\alpha P_5} = 0.236$

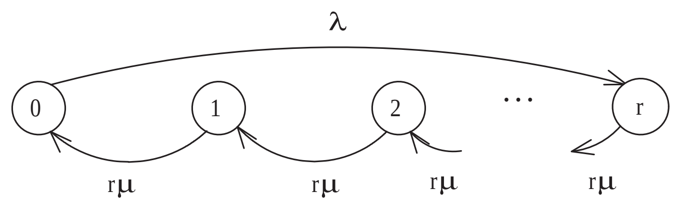
P(arrives in state 4) = $\frac{4\alpha P_4}{8\alpha P_0 + 7\alpha P_1 + 6\alpha P_2 + 5\alpha P_3 + 4\alpha P_4 + 3\alpha P_5} = 0.236$

P(W > 2h) = 0.0812

9. Stage method - the Erlang and Hyperexponential distributions

Ex. 9.1.

M/Er/1/1



a)

$$\lambda P_0 = r\mu P_1$$

$$r\mu P_1 = r\mu P_2$$

$$r\mu P_r = \lambda P_0$$

$$\Rightarrow \qquad P_0 = \frac{r\mu}{\lambda} P_1; \qquad P_1 = P_2 = \ldots P_r$$

$$\sum_{k=0}^{r} P_k = 1 \quad \Rightarrow \quad rP_1 + \frac{r\mu}{\lambda} P_1 = 1$$

$$P_0 = \frac{\mu}{\mu + \lambda}; \qquad P_1 = \frac{\lambda}{(\mu + \lambda) r}$$

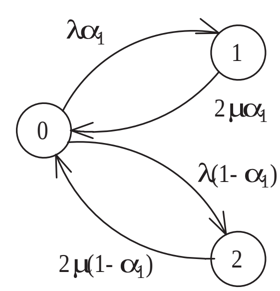
b)

$$P(\text{occupied}) = 1 - P_0 = \frac{\lambda}{\lambda + \mu}$$

Ex. 9.2.

$M/H_2/1/1$
$\mu_1 = 2\mu\alpha_1$
$\mu_2 = 2\mu(1 - \alpha_1)$

a)



$$\lambda\alpha_1 P_0 = 2\mu\alpha_1 P_1$$

$$\lambda(1 - \alpha_1) P_0 = 2\mu(1 - \alpha_1) P_2$$

$$\Rightarrow \qquad P_1 = \frac{\lambda}{2\mu} P_0; \qquad P_2 = \frac{\lambda}{2\mu} P_0$$

$$1 = P_0 \left(1 + \frac{\lambda}{2\mu} \cdot 2\right) = P_0 \left(1 + \frac{\lambda}{\mu}\right) \quad \Rightarrow \quad P_0 = \frac{\mu}{\lambda + \mu}$$

b)

$$P_1 = \frac{1}{2} \frac{\lambda}{\lambda + \mu}$$

c)

$$1 - P_0 = \frac{\lambda}{\lambda + \mu}$$

Ex. 9.3.

a)

$$\overline{W}_1 = \frac{1}{\mu} + \frac{1}{\mu} = 2 \text{ min}$$

$$\overline{W}_2 = \frac{1}{\mu} = 1 \text{ min}$$

b)

$$P(\text{wrong decision}) = P(W_2 > W_1) = \int_0^\infty P(W_2 > W_1 | W_1 = t) f_{W_1}(t) \mathrm{d}t$$

$$W_1 = X_1 + X_2, \quad X_1, X_2 \sim \text{Exp}(\mu) \qquad \Rightarrow \qquad W_1 \sim \text{Erlang}_2(\mu)$$

$$f_{W_1}(t) = \mu^2 t e^{-\mu t}$$

$$P(W_2 > W_1 | W_1 = t) = e^{-\mu t}$$

$$P(W_2 > W_1) = \int_0^\infty \mu^2 t e^{-\mu t} e^{-\mu t} \mathrm{d}t = \frac{1}{4}$$

c)

$$P(W_2 > W_1 | W_1 = t) = e^{-\frac{\mu t}{2}}$$

$$P(W_2 > W_1) = \int_0^\infty \mu^2 t e^{-\mu t} e^{-\frac{\mu t}{2}} \mathrm{d}t = \frac{4}{9}$$

Ex. 9.4.

$$\lambda = 3 \text{ msg/hour}; \quad \triangle t = 2 \text{ h}$$

a)

P(message will wait for more than 2 h)=P(no. of messages arriving in 2h < 4)

$$P(N = n) = \frac{(\lambda \triangle t)^n}{n!} e^{-\lambda \triangle t} \qquad \text{(Poisson process)}$$

$$P(N < 4) = \frac{(\lambda \triangle t)^3}{3!} e^{-\lambda \triangle t} + \frac{(\lambda \triangle t)^2}{2!} e^{-\lambda \triangle t} + \frac{(\lambda \triangle t)}{1!} e^{-\lambda \triangle t} + e^{-\lambda \triangle t} = 0.15$$

b)

$E[T]$ = expected time per message

$$E[T] = E[T|\text{short message}]P(\text{short message}) + E[T|\text{long message}]P(\text{long message}) =$$

$$= \frac{1000}{1200} \cdot \frac{3}{4} + \frac{10000}{1200} \cdot \frac{1}{4} = 2.708 \text{ sec}$$

$E[N]$ = expected number of messages per day

$$E[N] = \lambda \cdot 10 = 30 \text{ msg/day}$$

Per day one requires $\dfrac{E[N] \text{ msg/day}}{5 \text{ msg/connection}} = 6$ connections/day

Time per connection (5 messages) is $\quad 5 \cdot 2.708 + 7 = 20.54$ sec/connection

Expected cost is $\quad 20.54$ sec/connection $\cdot$ 6 connections/day $\cdot \dfrac{10 \cdot 20}{60}$ SEK/sec ≈ 21 SEK/day

Ex. 9.5.

The pdf of the Erlang-r distribution is: $\qquad f(x) = (\mu x)^{r-1} \mu e^{-\mu r x}$

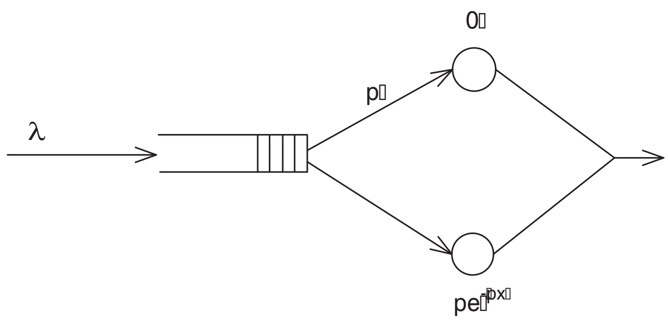
We look for the value of x for which $\mathrm{d}f/\mathrm{d}x = 0$.

$$f'(x) = (r\mu)^2 (r-1)(r\mu x)^{r-2} e^{-r\mu x} - (r\mu)^2 (r\mu x)^{r-1} e^{-r\mu x} =$$
$$= r\mu^2 e^{-r\mu x} (r\mu x)^{r-2} ((r-1) - r\mu x)$$

From $f'(a) = 0$ we get $r - 1 = r\mu a$. Therefore: $\qquad a = \dfrac{r-1}{r\mu} = \dfrac{1}{\mu} - \dfrac{1}{r\mu}$

10. M/G/1 queues

Ex. 10.1.



a)

$$\bar{S} = p\overline{S}_1 + (1-p)\overline{S}_2 = p \cdot 0 + (1-p)\frac{1}{p} = \frac{1-p}{p}$$

$$\overline{S^2} = p\overline{S_1^2} + (1-p)\overline{S_2^2} = p \cdot 0 + (1-p)\frac{2}{p^2} = \frac{2(1-p)}{p^2}$$

$$Var(S) = \sigma_s^2 = \overline{S^2} - \bar{S}^2 = \frac{1-p^2}{p^2}$$

b)

$$W = \frac{\lambda\overline{x^2}}{2(1-\rho)}; \qquad \rho = \lambda\bar{x} = \frac{\lambda(1-p)}{p} \qquad \Rightarrow \qquad W = \frac{\lambda \cdot \frac{2(1-p)}{p^2}}{2(1-\frac{\lambda(1-p)}{p})} = \frac{\rho}{p(1-\rho)}$$

$$S^*(s) = E[e^{-sS}] = E[e^{-s0}]p + E[e^{-sS_2}](1-p) = p + (1-p)\frac{p}{s+p} = \frac{p(s+1)}{s+p}$$

$$W^*(s) = \frac{s(1-\rho)}{s-\lambda+\lambda S^*(s)} = \cdots = \frac{(1-\rho)(s+p)}{s+p(1-\rho)}$$

c)

$$\overline{W} = -\frac{\mathrm{d}W^*(s)}{\mathrm{d}s}\Big|_{s=0} = -(1-\rho)\left[\frac{s+p(1-\rho)-(s+p)}{[s+p(1-\rho)]^2}\right]\Big|_{s=0} = \cdots = \frac{\rho}{p(1-\rho)}$$

$$W^*(s) = \frac{(1-\rho)(s+p)}{s+p(1-\rho)} = \cdots = 1-\rho+\rho\frac{p(1-\rho)}{s+p(1-\rho)}$$

$$f_W(t) = \pounds^{-1}[W^*(s)] = (1-\rho)\delta(t) + \rho\cdot(1-\rho)e^{-p(1-\rho)t}$$

$$F_W(t) = \int_0^t f_W(t)\mathrm{d}t = 1-\rho+\rho(1-e^{-p(1-\rho)t}) = 1-\rho e^{-p(1-\rho)t}$$

d)

$$G_N(z) = \frac{(1-\rho)(1-z)}{1-\frac{z}{S^*((1-z)\lambda)}} = ... = (1-\rho)\frac{(1-z)\lambda+1}{(1-\frac{z}{p})\lambda+1}$$

$$\overline{N} = \frac{\mathrm{d}G_N(z)}{\mathrm{d}z}\Big|_{z=1} = \lambda\frac{1-p(1-\rho)}{p(1-\rho)}$$

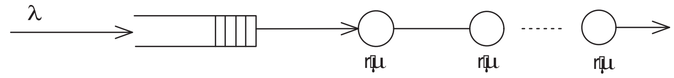
Ex. 10.2.

M/M/1:

$$\overline{N}_q = \overline{N} - \overline{N}_S = \frac{\rho}{1-\rho} - \rho = \frac{\rho^2}{1-\rho} \quad \Rightarrow \quad \rho = 0.9$$

a)

M/E$_5$/1



$$S = S_1 + S_2 + ... + S_5; \quad \overline{S}_i = \frac{1}{\mu}; \quad V[S_i] = \frac{1}{\mu^2}$$

$$\overline{S} = \sum_{i=1}^{5} \overline{S}_i \quad \Rightarrow \quad \overline{S} = \frac{5}{\mu}$$

$$V[S] = \sum_{i=1}^{5} V[S_i] \quad \Rightarrow \quad V[S] = \frac{5}{\mu^2}$$

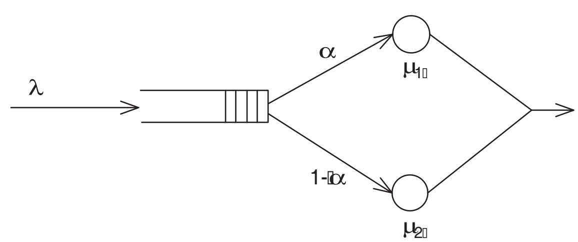
$$C_s^2 = \frac{V[S]}{\overline{S}^2} = \frac{1}{5}$$

$$\overline{N} = \rho + \frac{1 + C_s^2}{2} \frac{\rho^2}{1-\rho} = 5.76$$

If the number of stages $r \to \infty$, $C_s^2 \to 0$, M/E$_r$/1 $\to$ M/D/1

b)

M/H$_2$/1



$\mu_1, \mu_2 =?, \ \overline{N} = 15, \ \rho = 0.9, \ \alpha = 0.2, \ \lambda = 1800$ pkt/sec

$$\overline{S} = \frac{1}{\mu} = \frac{\rho}{\lambda} = \frac{1800}{0.9}$$

$$\overline{S}_1 = \frac{1}{\mu_1}, \qquad \overline{S}_2 = \frac{1}{\mu}$$

$$\overline{S} = \frac{1}{\mu} = \frac{1}{\mu_1}\alpha + \frac{1}{\mu_2}(1-\alpha) = \frac{0.2}{\mu_1} + \frac{0.8}{\mu_2}$$

$$\Rightarrow \quad \frac{0.2}{\mu_1} + \frac{0.8}{\mu_2} = \frac{1800}{0.9} \qquad (1)$$

$$\overline{N} = \rho + \frac{1+C_s^2}{2}\frac{\rho^2}{1-\rho} \quad \Rightarrow \quad C_s^2 = 2\frac{1-\rho}{\rho^2}(\overline{N}-\rho) - 1 = 2.4815$$

$$C_s^2 = \frac{\overline{S^2}}{\overline{S}^2} - 1 = 2.4815 \quad \Rightarrow \quad \overline{S^2} = 3.4815\left(\frac{0.9}{1800}\right)^2$$

$$\overline{S^2} = \frac{2}{\mu_1^2}\alpha + \frac{2}{\mu_2^2}(1-\alpha) = 2\left(\frac{0.2}{\mu_1^2} + \frac{0.8}{\mu_2^2}\right)$$

$$\frac{0.2}{\mu_1^2} + \frac{0.8}{\mu_2^2} = \frac{3.4815}{2}\left(\frac{0.9}{1800}\right)^2 \qquad (2)$$

From (1) and (2) we obtain:

$$\mu_1 = 735.3 \text{ pkt/sec}, \quad \mu_2 = 3508 \text{ pkt/sec}$$

Ex. 10.3.

a)

$$\lambda = \frac{1000}{120} \text{ pkt/sec}$$

$$\bar{x} = E[x_1] + E[x_2] + E[x_3] = 90 \text{ msec}$$

$$P_0 = 1 - \rho = 1 - \lambda\bar{x} = \frac{1}{4}$$

b)

$$\overline{N} = \rho + \rho^2\frac{1+C_x^2}{2(1-\rho)}, \qquad C_x^2 = \frac{\sigma_x^2}{\bar{x}^2}$$

$$X^*(s) = \left(\frac{\mu}{s+\mu}\right)^3 \quad \Rightarrow \quad \bar{x} = -X^{'*}(s)\Big|_{s=0} = \frac{3}{\mu}, \quad \overline{x^2} = X^{''*}(s)\Big|_{s=o} = \frac{12}{\mu^2}, \quad C_x^2 = \frac{1}{3}$$

$$\overline{N}_q = \overline{N} - \rho = \rho^2\frac{1+C_x^2}{2(1-\rho)} = \frac{3}{2}$$

Ex. 10.4.

a)

$$P(N=n) = P(\text{msg requires n transmissions}) = (1-p)p^{n-1} \quad \Rightarrow \quad N^{\sim} \text{ Geom}(1-p)$$

$$F_N(n) = P(N \le n) = \sum_{j=1}^{n} P(N=n) = 1 - p^n$$

b)

Service time is equal to the number of retransmissions $S = N$. We model the system as M/G/1 with service time distribution given in a).

$$\overline{T} = \overline{S} + \frac{\lambda\overline{S^2}}{2(1-\rho)}; \qquad \rho = \lambda\overline{S}$$

$$\overline{S} = \overline{N} = \frac{1}{1-p}, \quad \overline{S^2} = \sum_{n=0}^{\infty} n^2 P(N=n) = \frac{1+p}{(1-p)^2}$$

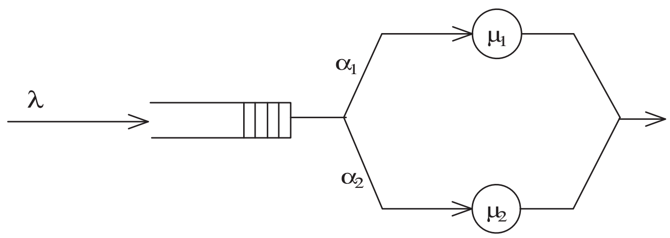
$$\Rightarrow \qquad T = \frac{1}{1-p} + \frac{\lambda}{2(1-p)}\frac{1+p}{(1-p)^2}$$

c)

$$\rho = \lambda\overline{S} < 1 \quad \Rightarrow \quad \frac{\lambda}{1-p} < 1 \quad \Rightarrow \quad \lambda < 1-p$$

Ex. 10.5.

$M/H_2/1$



a)

$$W = \frac{\rho\bar{x}(1+C_b^2)}{2(1-\rho)}; \qquad \rho = \lambda\bar{x}$$

$$B^*(s) = \frac{\alpha_1\mu_1}{s+\mu_1} + \frac{\alpha_2\mu_2}{s+\mu_2}$$

$$C_b^2 = \frac{\sigma_b^2}{\bar{x}^2} = \frac{\overline{x^2}}{\bar{x}^2} - 1 = \frac{350}{156} - 1 = 1.24$$

$$\bar{x} = 12.5 \text{ sec}; \qquad \rho = \lambda\bar{x} = \frac{5}{8}$$

$$\bar{x} = -B^{*'}(s)\Big|_{s=0} = \frac{\alpha_1}{\mu_1} + \frac{\alpha_2}{\mu_2}$$

$$\overline{x^2} = B^{*''}(s)\Big|_{s=0} = \frac{2\alpha_1}{\mu_1^2} + \frac{2\alpha_2}{\mu_2^2}$$

$$\Rightarrow \quad W = \frac{5}{8} \cdot \frac{50}{4} \cdot \frac{1+1.24}{1-\frac{5}{8}} \approx 23 \text{ sec}$$

b)

$$W^*(s) = \frac{s(1-\rho)}{s-\lambda+\lambda B^*(s)} = \frac{s(1-\rho)}{s-\lambda+\lambda\left[\frac{\alpha_1\mu_1}{s+\mu_1} + \frac{\alpha_2\mu_2}{s+\mu_2}\right]}$$

$$W^*(s) = (1-\rho)\left[1 + \frac{\frac{3}{4}}{s+\frac{9}{2}} + \frac{\frac{9}{4}}{s+\frac{3}{3}}\right]$$

$$w(y) = \mathcal{L}^{-1}\left[W^*(s)\right] = \frac{3}{8}\delta(y) + \frac{9}{32}e^{-\frac{9}{2}y} + \frac{27}{32}e^{-\frac{3}{2}y}; \qquad y \geq 0$$

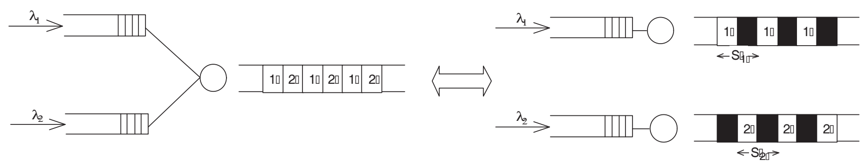
$$P(W > 2 \text{ min}) = \int_2^{\infty} w(y)\mathrm{d}y = 2.8\%$$

11. M/G/1 queues with vacation and priorities

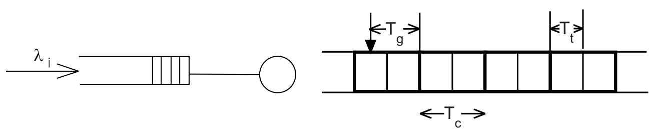
Ex. 11.1.

$\lambda_1 = 0.4$ msg/slot $\qquad \lambda_2 = 0.2$ msg/slot

length of one frame $T_c = 2$ slots



The equivalent queuing system for any of the two users



a)

$\overline{T}_g$: average time between a message arrival to the beginning of a new frame

$$T_g \sim \text{Uniform}(0, T_c) \quad \Rightarrow \quad \overline{T}_{g,1} = \overline{T}_{g,2} = \frac{T_c}{2} = 1 \text{ slot}$$

$\overline{T}_q$: average queuing time for an M/D/1, $\overline{S}_1 = \overline{S}_2 = T_c$

$$\overline{T}_{q,i} = \frac{\lambda_i \overline{S_i^2}}{2(1-\rho_i)}, \quad \rho_i = \lambda_i \overline{S}_i \quad \Rightarrow \quad \overline{T}_{q,1} = 4 \text{ slots}, \quad \overline{T}_{q,2} = \frac{2}{3} \text{ slots}$$

$\overline{T}_t$: message transmission time, $\overline{T}_{t,1} = \overline{T}_{t,2} = 1$ slot

$$\overline{T}_i = \overline{T}_{g,i} + \overline{T}_{q,i} + \overline{T}_{t,i} \quad \Rightarrow \quad \overline{T}_1 = 6 \text{ slots}, \quad \overline{T}_2 = \frac{8}{3} \text{ slots}$$

b)

$$P_1 = P_r(\text{msg from user 1}) = \frac{0.4}{0.4+0.2} = \frac{2}{3}$$

$$P_2 = P_r(\text{msg from user 2}) = \frac{0.2}{0.4+0.2} = \frac{1}{3}$$

$$\overline{T} = (\text{delay of a randomly chosen msg}) = \frac{2}{3} \cdot 6 + \frac{1}{3} \cdot \frac{8}{3} = \frac{44}{9} \approx 5 \text{ slots}$$

Ex. 11.2.

a)

T: slot time in A

$$D_A = T + \frac{\lambda T (10 \cdot T)}{2(1-\lambda T)} + \frac{10 \cdot T}{2}, \qquad D_B = 1.2 \cdot T + \frac{1.2 \lambda T (1.2 \cdot T)}{2(1-1.2\lambda T)} + \frac{1.2 \cdot T}{2}$$

B is better than A if $D_B < D_A \quad \Rightarrow \quad \lambda \leq \frac{0.73}{T}$

b)

Stable system:
$\rho < 1$ $\qquad \rho_A = \lambda_A T < 1 \quad \Rightarrow \quad \lambda_A < \frac{1}{T}\,; \qquad \rho_B < \frac{1}{1.2 \cdot T}$

Ex. 11.3.

$\lambda_A = 100$ pkt/sec, $\quad \overline{L}_A = 20$ bits

$\lambda_B = 20$ pkt/sec, $\quad \overline{L}_B = 100$ bits $\qquad C = 10000$ bits/sec

$$\overline{S}_A = \frac{20}{10 \cdot 10^3} = 2 \cdot 10^{-3} \qquad \overline{S_A^2} = 4 \cdot 10^{-6}$$

$$\overline{S}_B = \frac{100}{10 \cdot 10^3} = 10 \cdot 10^{-3} \qquad \overline{S_B^2} = 2 \cdot \overline{S}_B^2 = 200 \cdot 10^{-6}$$

$$\rho_A = 0.2, \qquad \rho_B = 0.2 \qquad \Rightarrow \qquad \rho = 0.2$$

$$\overline{R} = \frac{1}{2} \sum \lambda_i \overline{S_i^2} = \frac{1}{2}(400 \cdot 10^{-6} + 4000 \cdot 10^{-6}) = 2.2 \cdot 10^{-3}$$

a)

$A > B$:

$$\overline{W}_A = \frac{2.2 \cdot 10^{-3}}{1 - 0.2} = 2.75 \cdot 10^{-3} \text{ sec} \qquad \overline{T}_A = \overline{S}_A + \overline{W}_A = 4.75 \cdot 10^{-3} \text{ sec}$$

$$\overline{W}_B = \frac{2.2 \cdot 10^{-3}}{(1 - 0.2)(1 - 0.4)} = 4.58 \cdot 10^{-3} \text{ sec} \qquad \overline{T}_B = \overline{S}_B + \overline{W}_B = 14.58 \cdot 10^{-3} \text{ sec}$$

$A < B$:

$$\overline{W}_A = 4.58 \cdot 10^{-3} \text{ sec} \qquad \overline{T}_A = 6.58 \cdot 10^{-3} \text{ sec}$$

$$\overline{W}_B = 2.75 \cdot 10^{-3} \text{ sec} \qquad \overline{T}_B = 12.75 \cdot 10^{-3} \text{ sec}$$

b)

$A > B$:

$$\overline{R}_A = 200 \cdot 10^{-6}, \quad \overline{R}_B = R = 2.2 \cdot 10^{-3}$$

$$\overline{W}_A = 0.25 \cdot 10^{-3} \text{ sec} \qquad \overline{T}_A = \overline{S}_A + \overline{W}_A = 2.25 \cdot 10^{-3} \text{ sec}$$

$$\overline{W}_B = 4.58 \cdot 10^{-3} \text{ sec} \qquad \overline{T}_B = \overline{S'_B} + \overline{W}_B = 17.08 \cdot 10^{-3} \text{ sec}$$

$$\overline{S'_B} = \frac{\overline{S}_B}{1 - \rho_A} = \frac{10 \cdot 10^{-3}}{1 - 0.2} = 12.5 \cdot 10^{-3}$$

Ex. 11.4.

a)

$$C_h = 1000 * 800 = 800 kbits/s$$

$$C_{l,max} = 2Mbit/s - 800kbits/s = 1.2Mbit/s \qquad \lambda_{l,max} = C_{l,max}/L_l = 600 packet/s$$

b)

$$x_h = 0.4 * 10^{-3} s \qquad \overline{x}_l = 10^{-3} s$$

$$x_h{}^2 = 0.16 * 10^{-6} s \qquad \overline{x^2}_l = 2 * 10^{-6} s$$

If $\lambda_l = \lambda_{l,max}$ then: $W_h = \dfrac{R}{1-\rho_h} = ... = 1.13 * 10^{-3} s$

If $\lambda_l = 0$, then M/G/1: $W_h = 0.13 * 10^{-3} s$

c)

$\lambda_l = 300 pkts/s$

$$W_l = \frac{R}{(1-\rho_h)(1-\rho_h-\rho_l)} = 2.11 * 10^{-3} s$$

If the low priority buffer is never empty, like in part (b), then the high priory service can be modeled as M/G/1 with vacations.

Ex. 11.5.

$\lambda_1 = 60$ jobs/s $\qquad \lambda_2 = 40$ jobs/s

$\overline{S}_1 = 12 \times 10^{-3}$ s $\qquad \overline{S}_2 = 6 \times 10^{-3}$ s $\qquad \overline{V}_1 = 2 \times 10^{-3}$ s

a)

$$\overline{S} = \frac{\lambda_1}{\lambda_1+\lambda_2}\overline{S}_1 + \frac{\lambda_2}{\lambda_1+\lambda_2}\overline{S}_2 = 9.6 \times 10^{-3} \text{ s} \qquad \rho = (\lambda_1+\lambda_2) \times \overline{S} = 0.96$$

– with probability ρ, the server is busy: $\overline{R}_{v|\text{busy}} = 0$

– with probability $1-\rho$, the server is in maintenance: $\overline{R}_{v|\text{maintenance}} = \overline{V} = 2 \times 10^{-3}$

$$\overline{R}_v = \rho \overline{R}_{v|\text{busy}} + (1-\rho)\overline{R}_{v|\text{maintenance}} = 80\ \mu\text{s}$$

$$\overline{R}_{v|\text{empty}} = \overline{V} = 2 \times 10^{-3} \text{ s} \qquad \text{(memoryless of Exp)}$$

b)

$$\overline{T} = \overline{W} + \overline{S} \qquad \overline{W} = \frac{\lambda \overline{S^2}}{2(1-\rho)} + \frac{\overline{V^2}}{2\overline{V}}$$

$$\overline{S^2} = \frac{\lambda_1}{\lambda_1+\lambda_2}\overline{S_1^2} + \frac{\lambda_2}{\lambda_1+\lambda_2}\overline{S_2^2} = 0.6 \times (2\overline{S}_1^2) + 0.4 \times (2\overline{S}_2^2) = 201.6 \times 10^{-6}$$

$$\overline{V^2} = 2\overline{V}^2 = 8 \times 10^{-6}$$

$$\overline{W} = \frac{(\lambda_1+\lambda_2)\overline{S^2}}{2(1-\rho)} + \frac{\overline{V^2}}{2\overline{V}} = 0.254 \text{ s}$$

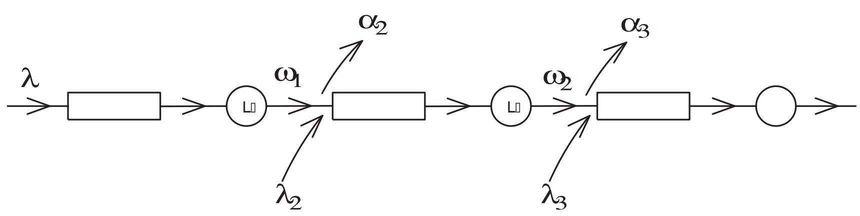
$$\overline{T} = \overline{W} + \overline{S} = 0.2636 \text{ s}$$

$$\overline{T}_1 = \overline{W} + \overline{S}_1 = 0.266 \text{ s}$$

$$\overline{T}_2 = \overline{W} + \overline{S}_2 = 0.26 \text{ s}$$

12. Queuing networks

Ex. 12.1.



a)

$$\rho_i = \frac{\lambda_{in}}{\mu_i}$$

$$\frac{1}{\mu_i} = \frac{1}{\mu} = \frac{1/\beta}{C}\frac{\text{bits}}{\text{bits/sec}} = \frac{1}{\beta C}\text{ sec} \qquad \Rightarrow \qquad \mu_i = \beta C$$

$$\rho_1 = \frac{\lambda}{\beta C}$$

$$\rho_2 = \frac{(\lambda + \lambda_2)(1-\alpha_2)}{\beta C}$$

$$\rho_3 = \frac{((\lambda + \lambda_2)(1-\alpha_2) + \lambda_3)(1-\alpha_3)}{\beta C}$$

b)

Every node is an M/M/1-system $\quad \Rightarrow \quad \overline{N}_i = \frac{\rho_i}{1-\rho_i}$

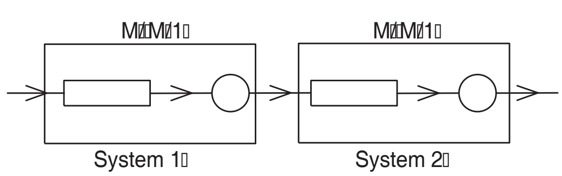
c)

$$\overline{T}_{tot} = \overline{T}_1 + \overline{T}_2 + \overline{T}_3$$

$$\overline{T}_i = \frac{\overline{N}_i}{\lambda_i} = \frac{1}{\mu_i(1-\rho_i)} = \frac{1}{\beta C(1-\rho_i)} \qquad \text{(Little's law)}$$

$$\overline{T}_{tot} = \frac{1}{\beta C}\sum_{i=1}^{3}\frac{1}{1-\rho_i}$$

Ex. 12.2.



$$\mu_1 = \mu_2 = \beta C$$

τ_{tot} time in the whole system $\qquad \tau_{tot} = \tau_1 + \tau_2 \quad \Rightarrow \quad S^*_{tot}(s) = S^*_1(s) + S^*_2(s)$

a)

$$T = T_1 + T_2$$

$$T^*(s) = T_1^*(s) T_2^*(s)$$

For M/M/1: $\quad T_i^*(s) = \dfrac{\mu - \lambda}{s + \mu - \lambda} \qquad$ (see note 1)

$$T^*(s) = \left(\frac{\beta C - \lambda}{s + \beta C - \lambda} \right)^2$$

$$f_T(t) = (\beta C - \lambda)^2 \cdot t \cdot e^{-(\beta C - \lambda)t}$$

b)

$$E(T) = -\frac{\mathrm{d}T^*(s)}{\mathrm{d}s}\bigg|_{s=0} = -(\beta C - \lambda)^2 \frac{-2}{(s + \beta C - \lambda)^3}\bigg|_{s=0} = \frac{2}{\beta C - \lambda}$$

$$E(T^2) = \frac{\mathrm{d}^2 T^*(s)}{\mathrm{d}s^2}\bigg|_{s=0} = (\beta C - \lambda)^2 \frac{6}{(s + \beta C - \lambda)^4}\bigg|_{s=0} = \frac{6}{(\beta C - \lambda)^2}$$

$$V(T) = E(T^2) - E(T)^2 = \frac{2}{(\beta C - \lambda)^2}$$

c)

$$P(T > t) = 1 - P(T \le t) = 1 - \int_0^t f_T(x)\mathrm{d}x = 1 - \int_0^t (\beta C - \lambda)^2 \cdot x \cdot e^{-(\beta C - \lambda)x}\mathrm{d}x =$$

$$= 1 - \left[-x \cdot (\beta C - \lambda) \cdot e^{-(\beta C - \lambda)x} \right]_0^t - \int_0^t (\beta C - \lambda) \cdot e^{-(\beta C - \lambda)x}\mathrm{d}x =$$

$$= 1 + t \cdot (\beta C - \lambda) \cdot e^{-(\beta C - \lambda)t} - \left[1 - e^{-(\beta C - \lambda)t} \right] = e^{-(\beta C - \lambda)t} \left[1 + (\beta C - \lambda)t \right]$$

Note 1:
$T^*(s)$ for a M/M/1 system:
If customer $k+1$ finds k customers in the system: $T_{k+1} = \sum_{i=1}^{k+1} X_i$

$$\Rightarrow T^*(s|k \text{ customers in the system}) = \left(\frac{\mu}{s + \mu} \right)^{k+1}$$

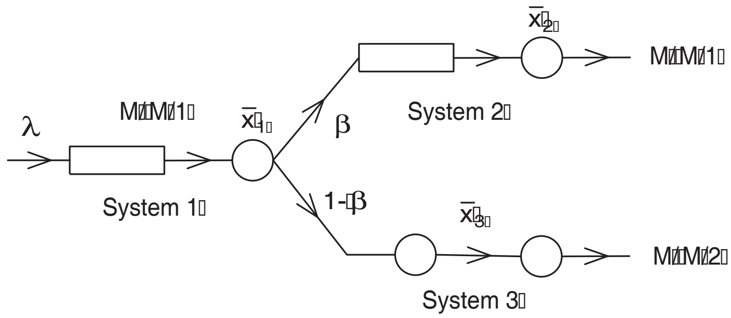
$$P_k = (1 - \rho)\rho^k$$

$$T^*(s) = \sum_{k=0}^{\infty} (1 - \rho)\rho^k \left(\frac{\mu}{s + \mu} \right)^{k+1} = (1 - \rho)\frac{\mu}{s + \mu} \sum_{k=0}^{\infty} \left(\frac{\lambda}{s + \mu} \right)^k =$$

$$= (1 - \rho)\frac{\mu}{s + \mu} \cdot \frac{1}{1 - \frac{\lambda}{s + \mu}} = \frac{\mu - \lambda}{s + \mu - \lambda}$$

Note 2:
Partial integral: $\quad \int u'v\mathrm{d}x = uv - \int uv'\mathrm{d}x$

Ex. 12.3.



$\lambda = 4$ jobs/min, $\bar{x}_1 = 10$ sec, $\bar{x}_2 = 30$ sec, $\bar{x}_3 = 20$ sec, $\beta = 0.2$

a)

$\lambda_2 = \lambda\beta = 0.8$ jobs/min $\lambda_3 = \lambda(1-\beta) = 3.2$ jobs/min

b)

$$\overline{N}_1 = \frac{\rho_1}{1-\rho_1} = \frac{\lambda\bar{x}_1}{1-\lambda\bar{x}_1} = 2 \qquad \overline{N}_2 = \frac{\rho_2}{1-\rho_2} = \frac{\lambda_2\bar{x}_2}{1-\lambda_2\bar{x}_2} = \frac{2}{3}$$

c)

Rejected jobs/min=$\lambda_3 E_m(\rho_3)$.

$\rho_3 = \lambda_3\bar{x}_3 = \lambda(1-\beta)\bar{x}_3 = 1.067$; $E_2(1.0) = 0.200000$ $E_2(1.1) = 0.223660$

Interpolation: $E_2(1.067) = 0.215852$.
Rejected jobs/min=$\lambda_3 E_2(1.067) = 0.67$/min.

d)

$$\overline{T} = \overline{T}_1 + \frac{\lambda\beta}{\lambda\beta + \lambda_3'} \cdot \overline{T}_2 + \frac{\lambda_3'}{\lambda\beta + \lambda_3'} \cdot \overline{T}_3$$

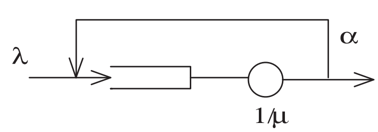
$$\overline{T}_1 = \frac{\bar{x}_1}{1-\rho_1}, \qquad \overline{T}_2 = \frac{\bar{x}_2}{1-\rho_2}, \qquad \overline{T}_3 = \bar{x}_3, \qquad \lambda_3' = \lambda(1-\beta)(1-E_m(\rho_3))$$

$\overline{T}_1 = 30$ sec, $\overline{T}_2 = 50$ sec, $\overline{T}_3 = 20$ sec, $\lambda_3' = 2.51 \Rightarrow$ $\overline{T} = 57.5$ sec

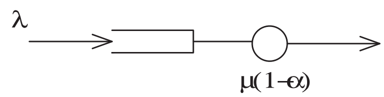
e)

$$\overline{W} = \overline{T}_1 - \bar{x}_1 + \frac{\lambda\beta}{\lambda\beta + \lambda_3'} \cdot (\overline{T}_2 - \bar{x}_2) = 24.8 \text{ sec}$$

Ex. 12.4.



a)
Service time distribution: jobs leave the system with intensity $\mu(1-\alpha)$.

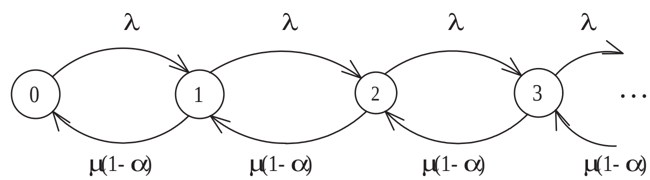


$$F_x(t) = 1 - e^{-\mu(1-\alpha)t}$$

b)

$$\bar{x} = \frac{1}{\mu(1-\alpha)}$$

c)



$$M/M/1 \text{ with } \rho = \frac{\lambda}{\mu(1-\alpha)} \quad \Rightarrow \quad P_k = \rho^k(1-\rho)$$

d)

$$\overline{N} = \frac{\rho}{1-\rho}$$

e)

$$\overline{W} = T - \bar{x} = \frac{\rho}{(1-\rho)\lambda} - \bar{x} = \frac{\bar{x}}{1-\rho} - \bar{x} = \frac{\rho\bar{x}}{1-\rho}$$

Ex. 12.5.

a)

$a_1 = \lambda$ $\qquad$ $a_5 = 0.5 \cdot \lambda + \lambda = 1.5 \cdot \lambda$

$a_3 = 0.5 \cdot a_5 + \lambda + 0.5 \cdot a_3 \quad \Rightarrow \qquad a_3 = 3.5 \cdot \lambda.$

$$a_2 = a_4 + \frac{1}{3}a_6 \quad a_4 = 0.5 \cdot \lambda + \frac{1}{3}a_6, \quad a_6 = a_2$$

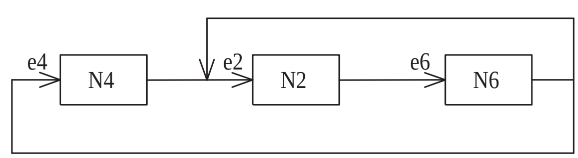
$$\Rightarrow \quad a_2 = 1.5 \cdot \lambda, \quad a_4 = \lambda, \quad a_6 = 1.5 \cdot \lambda$$

b)

$$\rho_1 = \frac{1}{4}, \quad \rho_2 = \frac{3}{8}, \quad \rho_3 = \frac{7}{8}, \quad \rho_4 = \frac{1}{4}, \quad \rho_5 = \frac{3}{8}, \quad \rho_6 = \frac{3}{8}$$

$$P_i^j = (1-\rho_j)\rho_j^i \qquad\qquad P(i_1, i_2, i_3, i_4, i_5, i_6) = \prod_{j=1}^{6} P_i^j$$

c)



$$e_2 = e_4 + e_6, \quad e_4 = 0.5 \cdot e_6, \quad e_6 = e_2 \qquad \Rightarrow \quad e_2 = e_6 = 2, \quad e_4 = 1$$

State			$G_3(3) \cdot P(\text{state})$	$P(\text{state})$
3	0	0	$\left(\frac{1}{\mu}\right)^3$	$\frac{1}{49}$
2	1	0	$\left(\frac{1}{\mu}\right)^2\left(\frac{2}{\mu}\right)$	$\frac{2}{49}$
2	0	1	$\left(\frac{1}{\mu}\right)^2\left(\frac{2}{\mu}\right)$	$\frac{2}{49}$
1	1	1	$\left(\frac{1}{\mu}\right)\left(\frac{2}{\mu}\right)^2$	$\frac{4}{49}$
1	2	0	$\left(\frac{1}{\mu}\right)\left(\frac{2}{\mu}\right)^2$	$\frac{4}{49}$
0	2	1	$\left(\frac{1}{\mu}\right)^3$	$\frac{8}{49}$
0	3	0	$\left(\frac{1}{\mu}\right)^3$	$\frac{8}{49}$
0	1	2	$\left(\frac{1}{\mu}\right)^3$	$\frac{8}{49}$
0	0	3	$\left(\frac{1}{\mu}\right)^3$	$\frac{8}{49}$
1	0	2	$\left(\frac{1}{\mu}\right)\left(\frac{2}{\mu}\right)^2$	$\frac{4}{49}$

$$G_3(3) = \frac{1+2+2+4+4+8+8+8+8+4}{\mu^3} = \frac{49}{\mu^3}$$

Solutions of exam problems

Ex. 1.

a)

$$P_{\text{loss}} \approx P_{K+1} = \rho^{K+1}(1-\rho) = 10^{-3}$$

$$\rho = \frac{\lambda}{\mu} = \frac{36 \cdot 10^3}{40 \cdot 10^3} = 0.9$$

$$\rho^{K+1} = \frac{10^{-3}}{1-0.9} \cdot 10^{-2} \qquad \Rightarrow \qquad K+1 = -2 \cdot \frac{\ln 10}{\ln 0.9} = 43.4 \quad \Rightarrow \quad K = 43$$

b)

$$\text{M/M/1/K:} \qquad P_{\text{loss}} = P_{K+1} = \frac{\rho^{K+1}(1-\rho)}{1-\rho^{K+2}} = 10^{-3}$$

$$\rho^{K+1} = \frac{10^{-3}}{1-0.9} \cdot (1-\rho^{K+2})$$

$$\rho^{K+1}(1 + 10^{-2} \cdot 0.9) = 10^{-2}$$

$$K+1 = m\frac{10^{-2}}{(1-10^{-2} \cdot 0.9) \cdot \ln 0.9} = 43.6$$

$$K = \lceil 43.6 \rceil - 1 = 43$$

Ratio of state probabilities for K+1:

$$\frac{\rho^{K+1}(1-\rho)}{\frac{\rho^{K+1}(1-\rho)}{1-\rho^{K+2}}} = 1 - \rho^{K+2} < 1$$

thus, the approximation underestimates the loss probability.

Ex. 2.

a)

$$P(X < x) = 1 - \left(\frac{a}{x}\right)^b, \qquad a, b > 0 \quad x > a$$

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x} P(X < x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(1 - \left(\frac{a}{x}\right)^b\right) = \frac{ba^b}{x^{b+1}}$$

$$C \cdot \bar{x} = \int_a^\infty f(x)\mathrm{d}x = \int_a^\infty x\frac{ba^b}{x^{b+1}}\mathrm{d}x = ba^b \int_a^\infty \frac{1}{x^b}\mathrm{d}x = \frac{ba^b}{b-1}\left[\frac{1}{x^{b-1}}\right]_a^\infty = \frac{ba}{b-1}$$

$$\rho = \lambda\bar{x} = 1500 \cdot \frac{250 \cdot 8}{8 \cdot 10^6}\frac{1}{b-1} = \frac{0.375}{b-1}$$

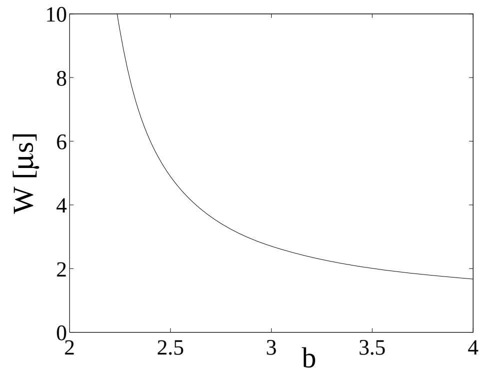
b)

$$W = \frac{\lambda\bar{x}}{2(1-\rho)}$$

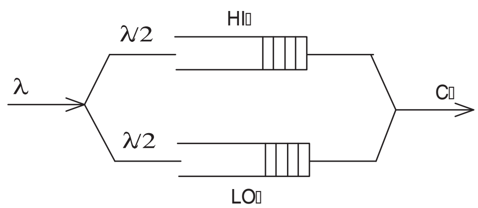
$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x} P(X < x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(1 - \left(\frac{a}{x}\right)^b\right) = \frac{ba^b}{x^{b+1}}$$

$$\overline{x^2} = \frac{1}{C^2}\int_a^\infty x^2 \frac{ba^b}{x^{b+1}}\mathrm{d}x = ba^b\int_a^\infty x^{-b+1}\mathrm{d}x = \frac{ba^b}{2-b}\left[\frac{1}{x^{b-2}}\right]_a^\infty = \begin{cases} \infty & b \le 2 \\ \frac{a^2 b}{b-2}\frac{1}{C^2} & b > 2 \end{cases}$$

$$W = \frac{\lambda}{C^2}\frac{a^2 b}{b-2}\frac{1}{2(1-\rho)} = 0.73\frac{b(b-1)}{(b-2)(b-1.38)}\ [\mu s]$$



c)



$$\text{Median file size} \quad \Rightarrow \quad P(X < x) = 0.5 \quad \Rightarrow \quad \left(\frac{a}{x}\right)^3 = 0.5 \quad \Rightarrow \quad x = a2^{1/3} = \frac{3}{5}\ \text{bytes}$$

Distribution of high priority packets:

$$P(\overline{X}_1 < x_1) = A\left(1 - \left(\frac{a}{x}\right)^3\right) \qquad a < x < a2^{1/3}$$

$A = 2$ (since it is half the original distribution)

$$C\overline{x}_1 = 2\int_a^{a2^{1/3}} x\frac{3a^3}{x^4}\mathrm{d}x = 3a^3\left[\frac{1}{x^2}\right]_{a2^{1/3}}^a = 3a(1 - 2^{-2/3}) = 1.11a = 277\ \text{bytes}$$

$$C^2\overline{x_1^2} = 2\int_a^{a2^{1/3}} x^2\frac{3a^3}{x^4}\mathrm{d}x = 6a^3\left[\frac{1}{x}\right]_{a2^{1/3}}^a = 6a^2(1 - 2^{-1/3}) = 1.24a^2$$

Distribution of low priority packets:

$$P(\overline{X}_2 < x_2) = 2\left(1 - \left(\frac{a}{x}\right)^3\right) \qquad a2^{1/3} < x < \infty$$

$$C\overline{x}_2 = 2\int_{a2^{1/3}}^\infty x\frac{3a^3}{x^4}\mathrm{d}x = 3a^3\left[\frac{1}{x^2}\right]_\infty^{a2^{1/3}} = 3a2^{-2/3} = 1.89a = 472\ \text{bytes}$$

$$C^2\overline{x_2^2} = 6a^3\left[\frac{1}{x}\right]_\infty^{a2^{1/3}} = 6a^2 2^{-1/3} = 4.76a^2$$

$$W_1 = \frac{R}{1-\rho_1} \qquad W_2 = \frac{R}{(1-\rho_1)(1-\rho_1-\rho_2)}$$

$$R = \frac{1}{2}\lambda_1\overline{x_1^2} + \frac{1}{2}\lambda_2\overline{x_2^2} = (\lambda_1 = \lambda_2 = \frac{\lambda}{2}) = \frac{1500}{4C^2}(1.24a^2 + 4.76a^2) = 141\,\mu s$$

$$W_1 = \frac{141}{1 - 750 \cdot \frac{277\cdot 8}{8\cdot 10^6}} = 178\,\mu s$$

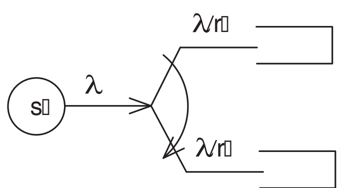
$$W_2 = \frac{141}{0.79(0.79 - \frac{730\cdot 472\cdot 8}{8\cdot 10^6})} = 406\,\mu s$$

Ex. 3.

	M/M/∞	birth-death
a)	$P_k = \frac{\rho^k}{k!}e^{-\rho}$	$P_k = \frac{\rho^k}{k!}e^{-\rho}$
b)	$\overline{N} = \rho$	$\overline{N} = \rho$
c)	$T = \frac{1}{\mu}$	$T = \frac{1}{\mu}$
d)	$W = 0$	$W = \frac{\overline{N}_q}{\lambda} = \frac{\rho-1+e^{-\rho}}{\lambda}$ $\overline{N}_q = \overline{N} - \overline{N}_s = \rho - 1 + e^{-\rho}$ $\overline{N}_s = 1 - P_0 = 1 - e^{-\rho}$
e)	$P(\text{wait}) = 0$	$P(\text{wait}) = 1 - P_0 = 1 - e^{-\rho}$
f)	offered load=$\rho = \frac{\lambda}{\mu}$	offered load=$\overline{N}_s = 1 - e^{-\rho}$

Ex. 4.

a)



Random: each stream Poisson with $VarX = \frac{1}{\lambda^2}$

Round-robin: each stream E_r-distributed with $VarX = \frac{1}{r\lambda^2} \to 0$ as $r \to \infty$

Round-robin is the most efficient since traffic gets less and less random as the number of paths increases.

b)

Undispersed system

$$\text{Total time} = T = \frac{\rho}{\lambda(1-\rho)}$$

$$\bar{x} = \frac{200 \cdot 8}{200 \cdot 10^3} = 8 \text{ ms}$$

$\lambda = 100$ pkt/sec, $\qquad \rho = \lambda\bar{x}$

$$T = \frac{100 \cdot 8}{100(1 - 100 \cdot 8 \cdot 10^{-3})} = 40 \text{ ms}$$

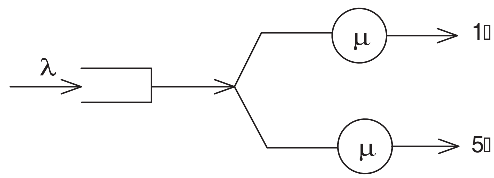
Dispersed system

$\lambda' = 20$ pkt/sec

$$T' = \frac{20 \cdot 40}{20(1 - 20 \cdot 40 \cdot 10^{-3})} = 200 \text{ ms}$$

$$\bar{x}' = \frac{200 \cdot 8}{40 \cdot 10^3} = 40 \text{ ms}$$

The undispersed system is more efficient (compared to random dispersion).
c)



M/M/k system

$$\lambda = 100 \text{ pkt/sec} \qquad \mu = \frac{40 \cdot 10^3}{200 \cdot 8} = 25 \text{ pkt/sec}$$

$$T = \frac{1}{\mu} + \frac{D_m(\rho)}{\mu(m-\rho)}, \qquad m = 5, \quad \rho = \frac{100}{25} = 4$$

$$D_m(\rho) = \frac{mE_m(\rho)}{m - \rho(1 - E_m(\rho))} \qquad E_5(4) = 0.199 \quad \Rightarrow \quad D_5(4) = \frac{5 \cdot 0.199}{5 - 4(1 - 0.199)} = 0.55$$

$$T = 40 + 40 \cdot \frac{D_5(4)}{5-4} = 40 \cdot (1 + 0.55) = 62 \text{ ms}$$

This system is less efficient than the undispersed system.

Ex. 5.

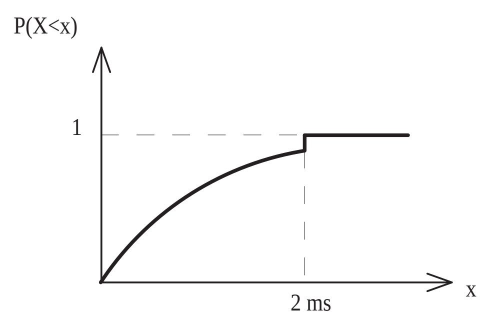
$$\lambda = 800 \text{ pkt/sec} \qquad \rho = \lambda \bar{x} = 800 \cdot 10^{-3} = 0.8$$

$$\bar{x} = \frac{8 \cdot 250 \text{ bytes}}{2 \cdot 10^6 \text{ bits/sec}} = 1 \text{ ms} \qquad \text{M/M/1 queue} \quad \Rightarrow \quad N = \frac{\rho}{1-\rho} = 4$$

a)

Truncated exponential distribution

$$F(x) = \begin{cases} B(1 - e^{-\mu x}) & x < 2 \text{ ms} \\ 1 & x \geq 2 \text{ ms} \end{cases}$$



$$\text{Find } \mu: \qquad \int_0^\infty f(x)\mathrm{d}x = 1 \quad \Rightarrow \quad B\mu \int_0^2 e^{-\mu x}\mathrm{d}x = 1 \quad \Rightarrow \quad 1 - e^{-2\mu} = \frac{1}{B} \quad \Rightarrow \quad B = 1.1565$$

$$N = \rho + \rho^2 \frac{1 + C_x^2}{2(1-\rho)}, \qquad C_x^2 = \frac{\sigma_x^2}{\bar{x}^2}$$

$$\bar{x} = B \int_0^2 x\mu e^{-\mu x}\mathrm{d}x = B\frac{e^{-\mu x}}{\mu}(\mu x + 1)\Big|_2^0 = 1.1565(1 - 0.4) = 0.694$$

Note that $\bar{x}$ cannot be 1 ms as stated above due to the truncation!

$$\overline{x^2} = B \int_0^2 x^2 \mu e^{-\mu x}\mathrm{d}x = e^{-\mu x}\left(\frac{x^2}{\mu} + \frac{2x}{\mu^2} + \frac{2}{\mu^3}\right)\Big|_2^0 = 1.1565(2 - 1.36) = 0.75$$

$$C_x^2 = \frac{\overline{x^2} - \bar{x}^2}{\bar{x}^2} = 0.5571$$

$$\rho = \lambda \bar{x} = 800 \cdot 0.694 \cdot 10^{-3} = 0.56$$

$$\overline{N} = 0.56 + 0.56^2 \frac{1 + 0.5571}{2(1 - 0.56)} = 1.12$$

The queue length is drastically reduced.

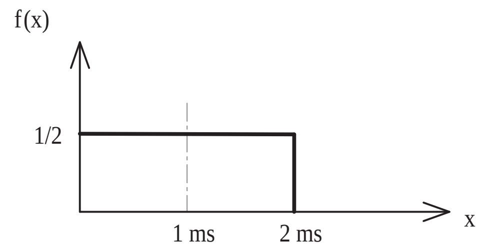
b)

$$\overline{x^2} = \int_0^2 x^2 \frac{1}{2} \mathrm{d}x = \frac{8}{6}$$

$$C_x^2 = \frac{8}{6} - 1 = \frac{1}{3}$$

$$\rho = \lambda \bar{x} = 800 \cdot 1 \cdot 10^{-3} = 0.8$$

$$\overline{N} = 0.8 + 0.8^2 \frac{1 + 1/3}{2(1 - 0.8)} = 2.93$$



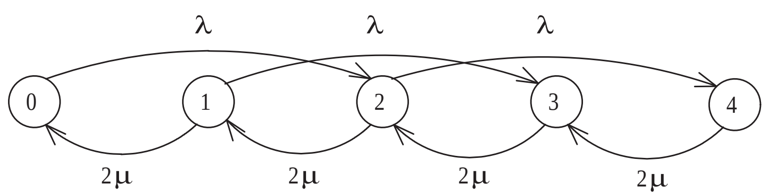
Uniform distribution of packet lengths gives shorter queue then exp. dist. at equal loads.

Ex. 6.

M/E2/1/2, λ=1 job/min, μ=1.5 job/min

a)

Arrival: Poisson
Service: two stage Erlang with $\tau = \frac{1}{2\mu}$, exponential service time
System capacity: one server, one buffer space



$$P_0 \lambda = P_1 2\mu$$

$$P_1(\lambda + 2\mu) = P_2 2\mu$$

$$P_2(\lambda + 2\mu) = P_0 \lambda + P_3 2\mu$$

$$P_3 2\mu = P_1 \lambda + P_4 2\mu$$

$$P_4 2\mu = P_2 \lambda$$

$$P_0 + P_1 + P_2 + P_3 + P_4 = 1$$

$$P_1 = P_0 \frac{\lambda}{2\mu}$$

$$P_2 = P_0 \frac{\lambda}{2\mu} \frac{\lambda + 2\mu}{2\mu}$$

$$P_3 = P_0 \left(\frac{\lambda}{2\mu}\right)^2 + \left(\frac{\lambda}{2\mu}\right)^2 \frac{\lambda + 2\mu}{2\mu}$$

$$P_4 = P_0 \left(\frac{\lambda}{2\mu}\right)^2 + \frac{\lambda + 2\mu}{2\mu}$$

$$\Rightarrow \quad P_0 = \frac{27}{53}, \quad P_1 = \frac{3}{59}, \quad P_2 = \frac{12}{59}, \quad P_3 = \frac{7}{59}, \quad P_4 = \frac{4}{59}$$

b)

$$P(\text{blocking}) = P(\text{system in states 3 or 4}) = P_3 + P_4 = \frac{11}{59}$$

$$T(\text{blocking}) = T(\text{service time}) = \frac{1}{2\mu} + \frac{1}{2\mu} = \frac{1}{\mu} = \frac{2}{8} \text{ min}$$

Poisson arrival with const. $\lambda \Rightarrow$ call blocking = time blocking

T(non-blocking) =?

$$P(\text{blocking}) = \frac{T(\text{blocking})}{T(\text{blocking}) + T(\text{non-blocking})} \quad \Rightarrow \quad T(\text{non-blocking}) = \frac{96}{33} \text{ min}$$

c)

T(mean time a job spends in the system)=?
Job is accepted in states 1,2, and 3.

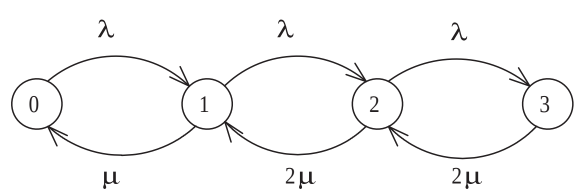
In state 0: $T_0 = \frac{1}{2\mu} + \frac{1}{2\mu} = \frac{1}{\mu}$ (the job's own service time)
In state 1: $T_1 = \frac{1}{2\mu} + \frac{1}{2\mu} + \frac{1}{2\mu} = \frac{3}{2\mu}$ (the second stage of the previous job $+T_0$)
In state 2: $T_2 = 4 \cdot \frac{1}{2\mu} = \frac{2}{\mu}$ (two stages of the previous job $+T_0$)

$$T = \frac{1}{P_0 + P_1 + P_2}\left(P_0 \cdot \frac{1}{\mu} + P_1 \cdot \frac{3}{2\mu} + P_2 \cdot \frac{2}{\mu}\right) = \frac{43}{48} \text{ sec}$$

Ex. 7.

M/M/2/3: λ=60 calls/min, $\bar{x} = \frac{1}{\mu}$=1.5 sec $\Rightarrow$ $\rho = 2$ Erlang

a)



b)

$$\lambda P_0 = \mu P_1 \quad \Rightarrow \quad P_1 = \rho P_0$$

$$\lambda P_2 = 2\mu P_3 \quad \Rightarrow \quad P_3 = \frac{\rho^3}{4} P_0$$

$$\lambda P_1 = 2\mu P_2 \quad \Rightarrow \quad P_2 = \frac{\rho^2}{2} P_0$$

$$P_0 + P_1 + P_2 + P_3 = 1 \quad \Rightarrow \quad P_0 = \frac{1}{1 + \rho + \frac{\rho^2}{2} + \frac{\rho^3}{4}} = \frac{1}{7}$$

$$P(\text{wait}) = P_2 = \frac{2}{7}$$

c)

$$W = W_0 P_0 + W_1 P_1 + W_2 P_2 + W_3 P_3 = W_2 P_2$$

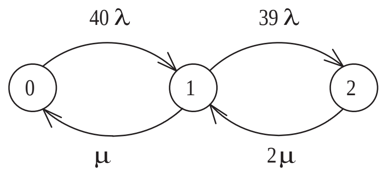
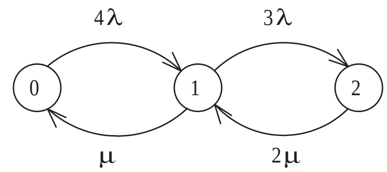
where W_k is the average waiting time when k customers are in the system.

$$W_2 = \frac{1}{2\mu}, \quad P_2 = \frac{2}{7} \quad \Rightarrow \quad W = \frac{2}{7} \text{ sec}$$

d)

$$\text{Mean blocking time} = \frac{1}{2\mu} = 1 \text{ sec}$$

Ex. 8.

<table>
<tr><th>A (Erlang)</th><th>B (Engset)</th></tr>
<tr><td>m=2
$\rho = \frac{40}{60} \cdot 6 = 4$ Erlang

</td><td>m=2
$a = \frac{\lambda}{\mu} = 0.1$


$4\lambda P_0 = \mu P_1 \quad \Rightarrow \quad P_1 = 0.4 \cdot P_0$
$3\lambda P_1 = 2\mu P_2 \quad \Rightarrow \quad P_2 = 0.06 \cdot P_0$
$P_0 = \frac{1}{1+0.46} = 0.685, \qquad P_1 = 0.274$</td></tr>
<tr><td>a)
P(blocking)=B(2,4)=(table)=0.615</td><td>a)
P(blocking)=P_2=0.041
Call blocking=$R_2 = \frac{2\lambda P_2}{4\lambda P_0 + 3\lambda P_1 + 2\lambda P_2} = 0.02256$</td></tr>
<tr><td>b)
Mean blocking time=$\frac{1}{2\mu} = 3$ sec</td><td>b)
Mean blocking time=$\frac{1}{2\mu} = 3$ sec</td></tr>
<tr><td>c)
Mean time w/o blocking* $= \frac{3(1-P(\text{blocking}))}{P(\text{blocking})} =$
$= 1.875$ sec</td><td>c)
Mean time w/o blocking* $= \frac{3(1-P_2)}{P_2} = 70$ sec</td></tr>
<tr><td>d)
$m = 8$ servers (6 extra)</td><td>d)
$m = 8$ servers (6 extra)</td></tr>
</table>

*Mean time w/o blocking $= \frac{\text{(Mean time w/o blocking)}}{\text{(Mean time with blocking)+(Mean time w/o blocking)}}$

Ex. 9.

$$P(L = 40 \text{ B}) = 0.65, \qquad P(L = 595 \text{ B}) = 0.20, \qquad P(L = 1500 \text{ B}) = 0.15$$

$$\lambda = 10^4 \text{ pkt/sec}, \qquad C = 34 \text{ Mbit¡s/sec}$$

a)

Average waiting time : $\quad W = \dfrac{\lambda \overline{x^2}}{2(1-\lambda\bar{x})}$

$$\overline{L} = 370 \text{ B} \quad \Rightarrow \quad \bar{x} = \frac{\overline{L}}{C} = 87 \cdot 10^{-6} \text{ sec}$$

$$E[x^2] = 0.65 \cdot \left(\frac{40 \cdot 8}{34 \cdot 10^6}\right)^2 + 0.20 \cdot \left(\frac{595 \cdot 8}{34 \cdot 10^6}\right)^2 + 0.15 \cdot \left(\frac{1500 \cdot 8}{34 \cdot 10^6}\right)^2 = 0.022 \cdot 10^{-6} \text{ sec}$$

$$\Rightarrow \qquad W = 0.846 \text{ msec}$$

b)

The transform function for the service time distribution:

$$B^*(s) = 0.65 \cdot e^{-\tau_1 s} + 0.20 \cdot e^{-\tau_2 s} + 0.15 \cdot e^{-\tau_3 s}$$

$$\tau_1 = \frac{L_1}{C} = 9.40 \ \mu\text{sec}; \quad \tau_2 = \frac{L_2}{C} = 0.14 \ \mu\text{sec}; \quad \tau_3 = \frac{L_3}{C} = 0.85 \ \mu\text{sec}$$

The transform function for the distribution of the number of packets in the node:

$$G(z) = B^*(\lambda - \lambda z) \cdot \frac{(1-\rho)(1-z)}{B^*(\lambda - \lambda z) - z}$$

$$B^*(\lambda - \lambda z) = 0.65 \cdot e^{-\tau_1(\lambda - \lambda z)} + 0.20 \cdot e^{-\tau_2(\lambda - \lambda z)} + 0.15 \cdot e^{-\tau_3(\lambda - \lambda z)}$$

$\rho = \lambda \bar{x} = 0.87$

c)

For exponentially distributed packet length (M/M/1):

$\bar{x} = 87 \cdot 10^{-6}$ sec

$\rho = \lambda \bar{x} = 0.87$

$$T = \frac{\rho}{1-\rho} = 0.669 \cdot 10^{-3} \text{ sec}$$

$$W = T - \bar{x} = (0.669 - 0.087) \text{ msec} = 0.582 \text{ msec}$$

The waiting time is lower in the exponential case because the second moment of the service time is lower ($E[x^2] = 2 \cdot \bar{x}^2 = 0.015 \cdot 10^{-6}$ sec).

Ex. 10.

M/M/1/S, $\rho = 0.5$, $S = K + 1$

a)

$$P(\text{blocking}) = P_S = \rho^S \frac{1-\rho}{1-\rho^{S+1}} = \frac{0.5^{K+1}(1-\rho)}{1-0.5^{K+2}}$$

$$\frac{0.5^{K+1}(1-\rho)}{1-0.5^{K+2}} \leq 0.0625 = 0.5^4 \qquad \Rightarrow \qquad 0.5^{K_{min}} = 0.5^2 - 0.5^4 \cdot 0.5^{K_{min}}$$

$$0.5^{K_{min}} = x \qquad \Rightarrow \qquad x = 0.5^2 - 0.5^4 x \qquad \Rightarrow \qquad x = 0.235294 \qquad \Rightarrow \qquad K_{min} = 3$$

b)

For M/M/1 system:

$$P(N > S) = \sum_{k=S+1}^{\infty} \rho^k (1-\rho) = (1-\rho)\left(\sum_0^{\infty} \rho^k - \sum_0^{S} \rho^k\right) = (1-\rho)\left(\frac{1}{1-\rho} - \frac{1-\rho^{S+1}}{1-\rho}\right) =$$

$$= \rho^{S+1} = 0.5^5 = 0.03125$$