Chapter 1 – Probability Theory and Transforms

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1 Exercise 1.2

X is a random variable chosen from X_1 with probability a and from X_2 with probability b. Calculate $E[X]$ and σ_X for $\alpha = 0.2$ and $b = 0.8$. X_1 is an exponentially distributed r.v. with parameter $\lambda_1 = 0.1$ and X_2 is an exponentially distributed r.v. with parameter $\lambda_2 = 0.02$. Let the r.v. Y be chosen from D_1 with probability α and from D_2 with probability b, where D_1 and D_2 are deterministic r.v.s. Calculate the values D_1 and D_2 so that $E[X] = E[Y]$ and $\sigma_X = \sigma_Y$.

Solution: a) We directly apply the **conditional expectation** formula:

$$E[X] = \alpha E[X_1] + bE[X_2].$$

We can do this since the expectation is a raw moment – not central. The proof is straightforward: we have

$$\begin{aligned} &f_X(x) = \alpha f_{X_1}(x) + b f_{X_2}(x) \rightarrow \\ &\rightarrow E[X] = \textstyle\int_0^\infty x f_X(x)dx = \alpha \int_0^\infty x f_{X_1}(x)dx + b\int_0^\infty x f_{X_2}(x)dx = \\ &\quad = \alpha E[X_1] + bE[X_2]. \end{aligned}$$

We then replace the given data

$$E[X] = \alpha\frac{1}{\lambda_1} + b\frac{1}{\lambda_2} = 0.2\frac{1}{0.1} + 0.8\frac{1}{0.02} = 42. \tag{1}$$

We can not calculate the variance (or the standard deviation) in the same way, since this is a central moment. Instead, we proceed with calculating the expected square of the r.v. X, which is a **raw** moment:

$$\begin{aligned} &E[X^2] = \textstyle\int_0^\infty x^2 f_X(x)dx = \alpha\int_0^\infty x^2 f_{X_1}(x)dx + b\int_0^\infty x^2 f_{X_2}(x)dx = \\ &\quad = \alpha E[X_1^2] + bE[X_2^2]. \end{aligned}$$

Replacing the data we get

$$E[X] = \alpha\frac{2}{\lambda_1^2} + b\frac{2}{\lambda_2^2} = 0.2\frac{2}{0.1^2} + 0.8\frac{2}{0.02^2} = 4040. \tag{2}$$

Finally, we use the relation between the expectation, square mean and variance

$$\sigma_X^2 = E[X^2] - [E[X]]^2 = 4040 - 42^2 \rightarrow \sigma_X = 47.70. \tag{3}$$

b) We have $E[Y] = \alpha d_1 + bd_2$ and $E[Y^2] = \alpha d_1^2 + bd_2^2$. So the system of equations becomes

$$\begin{aligned} 0.2d_1 + 0.8d_2 &= 42, \\ 0.2d_1^2 + 0.8d_2^2 &= 4040 \end{aligned} \tag{4}$$

Solving this 2 by 2 non-linear system we obtain the solution. Notice that because of the second order of the equation we may have more than one solutions.

2 Exercise 1.3

X is a discrete stochastic variable, $p_k = P(X = k) = \frac{a^k}{k!}e^{-a}, k = 0, 1, 2, ...$ and a is a positive constant.

a) Prove that $\sum_{k=0}^{\infty} p_k = 1$.

b) Determine the z-transform (generating function) $P(z) = \sum_{k=0}^{\infty} z^k p_k$.

c) Calculate $E[X], \mathrm{Var}[X]$ and $E[X(X-1)...(X-r+1)], r = 1, 2, ...$ with and without using z-transforms.

Solution a) We have

$$\sum_{k=0}^{\infty} p_k = \sum_{k=0}^{\infty} \frac{a^k}{k!}e^{-a} = e^{-a}\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^{-a}e^{a} = 1.$$

Notice this useful and well-known infinite series summation.

b) We replace the definition of the mass function and gradually have:

$$P(z) = \sum_{k=0}^{\infty} z^k \frac{a^k}{k!}e^{-a} = e^{-a}\sum_{k=0}^{\infty} z^k \frac{a^k}{k!} = e^{-a}\sum_{k=0}^{\infty} \frac{(za)^k}{k!} = e^{-a}e^{az} = e^{-a(1-z)}.$$

c) First, we try without the z-transform, i.e. using the definitions in the probability domain. We start from the third sentence, using the **definition** of expectation:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx \tag{5}$$

$$\begin{aligned} E[X(X-1)...(X-r+1)] &= \textstyle\sum_{k=0}^{\infty} k(k-1)...(k-r+1)p_k = \\ &= \textstyle\sum_{k=0}^{\infty} k(k-1)...(k-r+1)\frac{a^k}{k!}e^{-a} = \sum_{k=0}^{\infty} \frac{a^k}{(k-r)!}e^{-a} = \\ &= e^{-a}a^r \textstyle\sum_{k=0}^{\infty} \frac{a^{(k-r)}}{(k-r)!} = a^r e^{-a}e^{a} = a^r. \end{aligned}$$

Then clearly, we have (by setting $r = 1$) $E[X] = a^1 = a$. And, finally,

$$\mathrm{Var}[X] = E[X^2] - [E[X]]^2 = E[X^2] - a^2 = E[X(X-1)] + E[X] - a^2 = a^2 + a - a^2.$$

We try, now, with the z-transform. We differentiate r times the definition of the z-transform:

$$\frac{d^r}{dz^r}P(z) = \frac{d^r}{dz^r}\sum_{k=0}^{\infty} z^k p_k = \sum_{k=0}^{\infty} k(k-1)...(k-r+1)z^{k-r}p_k$$

If we replace $z = 1$ we get

$$\left.\frac{d^r}{dz^r}P(z)\right\}_{z=1} = E[X(X-1)...(X-r+1)].$$

We, then, calculate,

$$\left.\frac{d^r}{dz^r}P(z)\right\}_{z=1} = a^r e^{-a(1-1)} = a^r.$$

3 Exercise 1.4

X_i's are independent Poisson distributed random variables, thus, $p_k = \frac{a_i^k}{k!}e^{-a_i}$, $k = 0, 1, 2, ...,$ and each $a_i, i = 1, 2, ..., n$ is a positive constant. Give the probability distribution function of $X = \sum_{i=1}^{n}$.

Solution: This problem indicates the usefulness of the z-transform in the calculation of the distribution of the sum of variables. We have proven that **the ZT of the sum of independent random variables is the product of their individual z-transforms**. Thus,

$$P(z) = \prod_{i=1}^{n} P_i(z) = \prod_{i=1}^{n} e^{-a_i(1-z)} = e^{\sum_{i=1}^{n} -a_i(1-z)} = e^{-\alpha(1-z)},$$

where $\alpha = \sum_{i=1}^{n} -a_i$. This proves that the distribution is also Poisson with parameter α, i.e. the sum of parameters. The proof is based on the uniqueness of z-transform[1]. As a result, the distribution function will be

$$p_X(k) = \frac{\alpha^k}{k!}e^{-\alpha}$$

4 Exercise 1.5

X is a positive stochastic continuous variable with probability distribution function (PDF)

$$F(x) = P(X \leq x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-ax}, x \geq 0. \end{cases}$$

a) Give the probability density function $f(x) = dF(x)/dx$.

b) Give $\overline{F}(x) = P(X > x)$.

c) Calculate the Laplace Transform $f^*(s) = E[e^{-sX}] = \int_0^\infty e^{-sx}f(x)dx$.

d) Calculate the expected values $m = E[X], E[X^k]$, $k = 0, 1, 2, ...,$ the variance σ_X^2, the standard deviation σ_X and the coefficient of variation $c = \sigma/m$, with and without the transform $F^*(s)$.

Solution: a) For the calculation of $f(x)$ we just need to differentiate:

$$f(x) = dF(x)/dx = d(1 - e^{-ax})/dx = ae^{-ax}.$$

b) The complementary PDF is simply given as

$$\overline{F}_X(x) = P(X > x) = 1 - P(X \leq x) = 1 - F_X(x) = e^{-ax}.$$

[1]or the 1-1 correspondence between the mass function and the ZT

c) Calculation of the Laplace Transform with simple integration

$$f^*(s) = \int\limits_0^\infty e^{-sx} f(x) dx = \int\limits_0^\infty e^{-sx} a e^{-ax} dx = a \int\limits_0^\infty e^{-x(s+a)} dx = \frac{a}{s+a}.$$

d) We proceed first, without the help of Laplace transforms, using the definition of the expectation

$$E[X^0] = \int_0^\infty x^0 f(x) dx = \int_0^\infty f(x) dx = 1.$$

$$\begin{aligned} E[X^k] &= \int_0^\infty x^k f(x) dx = \int_0^\infty x^k a e^{-ax} dx = a \tfrac{-1}{a} \int_0^\infty x^k (e^{-ax})' dx = \\ &= k \int_0^\infty x^{k-1} e^{-ax} dx = \tfrac{k}{a} \int_0^\infty x^{k-1} a e^{-ax} dx = \int_0^\infty x^{k-1} f(x) dx = \\ &= \tfrac{k}{a} E[X^{k-1}]. \end{aligned}$$

This is a recursive formula that enables the calculation of any moment. We have:

$$E[X^k] = \frac{k}{a} E[X^{k-1}] = \frac{k}{a} \frac{k-1}{a} E[X^{k-2}] = \frac{k}{a} \frac{k-1}{a} .. \frac{1}{a} E[X^0] = \frac{k}{a} \frac{k-1}{a} .. \frac{1}{a} = \frac{k!}{a^k}$$

which gives, simply, $E[X] = 1/a$, for $k = 1$. The variance is calculated through the usual formula, and the raw moments are taken from above:

$$\sigma^2 = E[X^2] - [E[X]]^2 = \frac{2}{a^2} - \left(\frac{1}{a}\right)^2 = 1/a^2.$$

so the standard deviation is simply the square root of the variance, $1/a$, and the coefficient of variation is 1. Notice that this is special for the exponential distribution.

We try, now, with the help of the Laplace transforms.

$$E[X^k] = (-1)^k \tfrac{d^k}{ds^k} f^*(s) = (-1)^k \tfrac{d^k}{ds^k} \tfrac{a}{s+a} = \tfrac{(-1)^k a k!}{(s+a)^{k+1}}.$$

We find this formula by differentiating k times the Laplace transform and replacing $s = 0$. The rest follows with simple replacement $k = 1, 2, ...$

5 Exercise 1.6

X_i's are independent, exponentially distributed random variables with a mean value of $1/a, a > 0, i = 1, 2, ..., n$. Calculate $P(X \leq x)$ and $P(X \geq x)$ where

a) $X = \min(X_1, X_2, ..., X_n)$,

b) $X = \max(X_1, X_2, ..., X_n)$.

Solution: a) The key point in this exercise is the fact that the random variables are independent (mutually independent). We gradually have:

$$\begin{aligned} P(X \leq x) &= P(\min(X_1, X_2, ..., X_n) \leq x) = 1 - P(\min(X_1, X_2, ..., X_n) > x) \\ &= 1 - P(X_1 > x, X_2 > x, ..., X_n > x) = 1 - \textstyle\prod_{i=1}^n P(X_i > x) \\ &= 1 - \textstyle\prod_{i=1}^n e^{-ax} = 1 - e^{-\sum_{i=1}^n ax} = 1 - e^{-nax} \end{aligned}$$

This shows that the minimum of exponentially distributed random variables is also an exponential variable and its rate is the sum of the individual rates.

b) Similar calculations:

$$P(X \leq x) = P(\max(X_1, X_2, ..., X_n) \leq x) = P(X_1 \leq x, X_2 \leq x, ..., X_n \leq x)$$
$$= \prod_{i=1}^{n} P(X_i \leq x) = \prod_{i=1}^{n} (1 - e^{-ax}) = (1 - e^{-ax})^n.$$

Cleary, the variable X is, now, not exponential.