DFT AND IDFT

DFT:

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}} \quad k = 0, 1, \ldots, N-1$$

IDFT:

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi kn}{N}} \quad n = 0, 1, \ldots, N-1$$

RELATIONSHIP DFT $\leftrightarrow$ DTFT

$$X(k) = X_N(f)\rfloor_{f=\frac{k}{N}}$$

- Truncation of the signal to length N:

$$x_N(n) = \begin{cases} x(n) & n = 0, 1, \ldots, N-1 \\ 0 & \text{otherwise} \end{cases}$$

- Sampling of the frequency axis:

$$f = \frac{k}{N}$$

Zero Padding

If $x(n)$ has length N and we want to evaluate

$$X_N(f) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi fn}$$

at $M > N$ frequency values, calculate the M-point DFT of

$$x_{\text{ZP}}(n) = \{x(0), x(1), \ldots, x(N-1), \underbrace{0, \ldots, 0}_{M-N \text{ zeros}}\}$$

Gives

$$X_{\text{ZP}}(k) = X_N(f)\rfloor_{f=\frac{k}{M}}$$

Matrix formulation of DFT

DFT: $\mathbf{X} = \mathbf{W}\mathbf{x}$

IDFT: $\mathbf{x} = \frac{1}{N}\mathbf{W}^H\mathbf{X}$

where ($w_N = e^{-\frac{j2\pi}{N}}$)

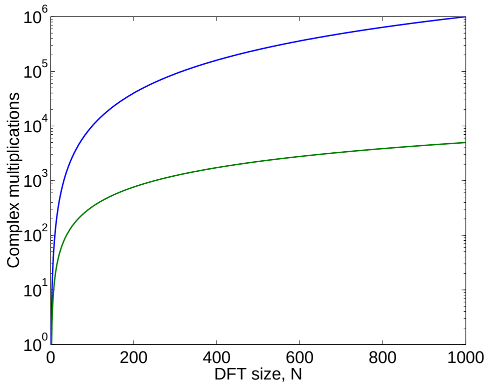
$$\mathbf{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & w_N & \cdots & w_N^{N-1} \\ \vdots & & \ddots & \vdots \\ 1 & w_N^{N-1} & \ddots & w_N^{(N-1)(N-1)} \end{bmatrix}$$

$$\frac{1}{\sqrt{N}}\mathbf{W} \text{ is a unitary (orthogonal) matrix } \iff \left(\frac{1}{\sqrt{N}}\mathbf{W}\right)^{-1} = \frac{1}{\sqrt{N}}\mathbf{W}^H$$

FAST FOURIER TRANSFORM (FFT)

FFT is a *fast algorithm* for computing the DFT.

	Direct computation	Radix-2 FFT
Complex multiplications	N^2	$\frac{N}{2}\log_2 N$
Order of complexity	$O(N^2)$	$O(N\log_2 N)$



FFT (RADIX-2) OBSERVATION

- Length N sequence $x(n)$, $\quad X(k) = \mathrm{FFT}_N[x(n)]$
  - even elements: $x_e(m) = x(2m)$, $\quad X_e(k) = \mathrm{FFT}_{N/2}[x_e(m)]$
  - odd elements: $x_o(m) = x(2m+1)$, $\quad X_o(k) = \mathrm{FFT}_{N/2}[x_o(m)]$

$$\Rightarrow X(k) = X_e(k) + e^{-j2\pi\frac{k}{N}} X_o(k)$$

REMARKS, FFT

- Several different kinds of FFTs! These provide trade-offs between multiplications, additions and memory usage.
- Other important aspects are parallel computation, quantization effects and bit representation in each stage.
- Renewed interest in FFT algorithms due to OFDM (Orthogonal Frequency Division Multiplexing) used in ADSL, Wireless LAN, 4G wireless (LTE) and digital radio broadcast (DAB).