Chapter 3 – Balance equations, birth-death processes, continuous Markov Chains

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1 Exercise 3.2

Consider a birth-death process with 3 states, where the transition rate from state 2 to state 1 is $q_{21} = \mu$ *and* $q_{23} = \lambda$*. Show that the mean time spent in state 2 is exponentially distributed with mean* $1/(\lambda + \mu)$.[1]

Solution: Suppose that the system has just arrived at state 2. The time until next "birth" – denoted here as T_B – is exponentially distributed with cumulative distribution function $F_{T_B}(t) = 1 - e^{-\lambda t}$. Similarly, the time until next "death" – denoted here as T_D – is exponentially distributed with cumulative distribution function $F_{T_D}(t) = 1 - e^{-\mu t}$. The random variables T_B and T_D are independent.

Denote by T_2 the time the system spends in state 2. The system will depart from state 2 when the first of the two events (birth or death) occurs. Consequently we have $T_2 = \min\{T_B, T_D\}$. We, then, apply the result from exercise 1.6, that is the minimum of independent exponential random variables is an exponential random variable. We can actually show this:

$$
\begin{aligned}
F_{T_2}(t) &= \Pr\{T_2 \le t\} = \\
&= \Pr\{\min\{T_B, T_D\} \le t\} = \\
&= 1 - \Pr\{\min\{T_B, T_D\} > t\} = \\
&= 1 - \Pr\{T_B > t, T_D > t\} = \\
&= 1 - \Pr\{T_B > t\} \cdot \Pr\{T_D > t\} = \\
&= 1 - e^{-\lambda t} \cdot e^{-\mu t} = \\
&= 1 - e^{-(\lambda+\mu)t}
\end{aligned}
$$

so T_2 is exponentially distributed with parameter $\lambda + \mu$.

Notice that we can generalize to the case with more than two transition branches. This exercise reveals the property of continuous time Markov chains, that is, the time spent on a state is exponentially distributed.

2 Exercise 3.3

Assume that the number of call arrivals between two locations has Poisson distribution with intensity λ*. Also, assume that the holding times of the conversations*

[1] This exercise is similar to Exercise 6 from Chapter 1: "The minimum of independent exponential variables is exponential."

are exponentially distributed with a mean of $1/\mu$*. Calculate the average number of call arrivals for a period of a conversation.*

Solution: Denote by N_C the number of arriving calls during the period of one conversation. Denote by T the duration of this conversation. Given that $T = t$, $N_C|T = t$ is Poisson distributed with parameter $\lambda \cdot t$ so the probability mass function of the number of calls will be

$$\Pr\{\text{arriving calls within } t = k\} = P_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

with an average number of calls: $E[N_C|T = t] = \lambda t$.

Moreover T is exponentially distributed, with parameter μ so the density function will be:

$$f_T(t) = \mu e^{-\mu t}.$$

We apply the conditional expectation formula:

$$E[N_C] = \int_0^\infty E[N_C|T = t] \cdot f_T(t) dt = \int_0^\infty \lambda t \mu e^{-\mu t} dt = \lambda \int_0^\infty t \mu e^{-\mu t} dt = \frac{\lambda}{\mu}.$$

3 Exercise 3.4

Consider a communication link with a constant rate of 4.8*kbit/sec. Over the link we transmit two types of messages, both of exponentially distributed size. Messages arrive in a Poisson fashion with* $\lambda = 10$ *messages/second. With probability 0.5 (independent from previous arrivals) the arriving message is of type 1 and has a mean length of 300 bits. Otherwise a message of type 2 arrives with a mean length of 150 bits. The buffer at the link can at most hold one message of type 1 or two messages of type 2. A message being transmitted still takes a place in the buffer.*

a) Determine the mean and the coefficient of variation of the service time of a randomly chosen arriving message.

b) Determine the average times in the system for accepted messages of type 1 and 2.

c) Determine the message loss probabilities for messages of type 1 and 2.

Solution:

a) We have a link with a constant transmission rate. So the service time distributions follow the packet length distributions. Consequently, the service times of both packet types are exponential with mean values of

- Type 1: $E[T_1] = \frac{300}{4800} = \frac{1}{16}$ sec,
- Type 2: $E[T_2] = \frac{150}{4800} = \frac{1}{32}$ sec.

As a result the parameters of the exponential distributions are $\mu_1 = 16$ and $\mu_2 = 32$, respectively. A random arriving packet is of Type 1 or Type 2 with probability 0.5. We apply the conditional expectation [2]:

$$E[T] = \frac{1}{2} E[T_1] + \frac{1}{2} E[T_2] = \frac{3}{64}$$

[2] This is similar to exercise 1.2

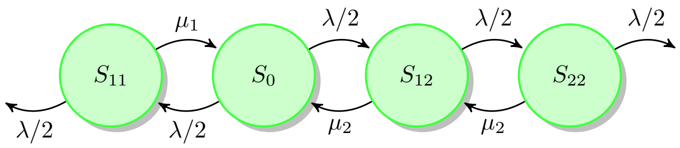

Figure 1: State Diagram for Exercise 3.4

Similarly, we calculate the mean square:

$$E[T^2] = \frac{1}{2}E[T_1^2] + \frac{1}{2}E[T_2^2] = \frac{1}{2}\frac{2}{\mu_1^2} + \frac{1}{2}\frac{2}{\mu_2^2} = 16^{-2} + 32^{-2} = \frac{5}{4} \cdot 16^{-2}.$$

The variance of T is derived from $Var[T] = E[T^2] - (E[T])^2$. Then we compute the standard deviation σ_T as $\sigma_T = \sqrt{Var[T]}$, and finally the coefficient of variation is given as: $c_T = \frac{\sigma_T}{E[T]}$.

b) For this part of the exercise, we need to draw the Markov Chain (Fig. 1) and solve it in the **steady state**. The state space must be defined in such a way that we can guarantee that all transitions – from state to state – have an exponential rate. We choose here to define such a Markov chain with 4 states:
State 0; Empty buffer.
State 11; 1 packet of Type 1.
State 21; 1 packet of Type 2.
State 22; 2 packets of Type 2.
Then we solve the balance equations in the local form:

$$\begin{aligned}
\mu_2 P_{22} &= \lambda/2 P_{21} \\
\mu_2 P_{21} &= \lambda/2 P_0 \\
\mu_1 P_{11} &= \lambda/2 P_0 \\
P_0 + P_{11} + P_{21} + P_{22} &= 1 (\text{norm. equation})
\end{aligned}$$

Solution:

$$P_0 = 0.670, P_{21} = 0.105, P_{22} = 0.016, P_{11} = 0.209.$$

An accepted message of Type 1 can only arrive at state 0, otherwise it is rejected. So its the average service time will be $E[T_1]$.
An accepted message of Type 2 can arrive at states 0 and 21, otherwise it is rejected. Then, the average service time will be $(E[T_2]P_0 + 2E[T_2]P_{(21)})/(P_0 + P_{21})$. [3]

c) The loss probabilities are equal to the probabilities of the system being in BLOCKING states, for each of the two packet types. We underline that this is always true for *homogeneous* Markov chains, that is, Markov chains where the arrival rates do not depend on the system state.

[3] The occurrence of acceptance reduces the sample space to two states only. Then the probabilities are normalized.

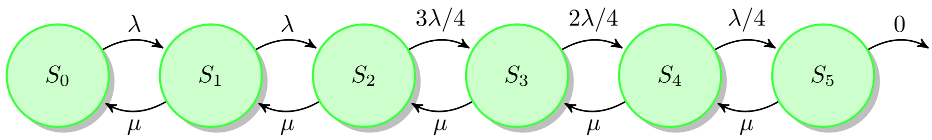

Figure 2: State diagram for Exercise 3.5

4 Exercise 3.5

Consider a Markovian system with discouraged job arrivals. Jobs arrive to a server in a Poisson fashion, with an intensity of one job per 7 seconds. The jobs observe the queue. They do NOT join the queue with probability l_k if they observe k jobs in the queue. $l_k = k/4$ if $k < 4$, or 0, otherwise. The service time is exponentially distributed with mean time of 6 seconds.

a) Determine the mean number of customers in the system, and

b) the number of jobs served in 100 seconds.

Solution:

a) This is a simple model but requires careful design. After building the correct state diagram, the solution is found, based on the LOCAL balance equations.

We have a system with 6 states. *State space*: S_k : k jobs in the system. The system diagram is shown in Fig. 2).

Balance Equation System:

$$\lambda P_0 = \mu P_1$$
$$\lambda P_1 = \mu P_2$$
$$3\lambda/4 P_2 = \mu P_3$$
$$\lambda/2 P_2 = \mu P_4$$
$$\lambda/4 P_2 = \mu P_5$$
$$\sum_{k=1}^{5} P_k = 1$$

Solution: $P_0 \approx 0.3$, and the remaining probabilities are computed based on P_0 and the equations above. After determining the state probabilities, we derive the average number of customers in the system through

$$E[N] = \sum_{k=0}^{5} k \cdot P_k$$

We find $E[N] \approx 1.43$.

b) We have, here, a system with different arrival rates in each state. These systems are defined as *non-homogeneous*. However, the service rate is constant. The server is busy with probability $(1 - P_0)$. When it is busy, it serves jobs. The service rate is $\mu = 1/6 sec^{-1}$. As a result, the server can serve $100 \cdot \mu \cdot (1 - P_0)$ jobs in 100 seconds on AVERAGE!

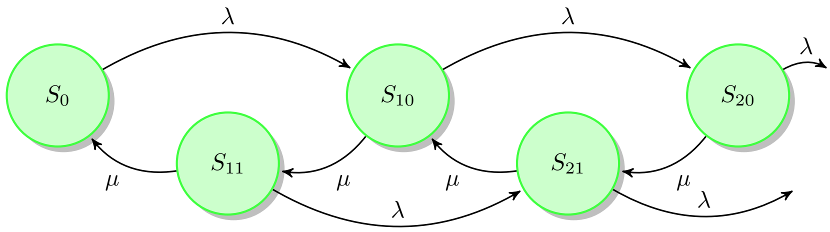

Figure 3: State Diagram for Exercise 3.6

5 Exercise 3.6

Consider a network node that can serve 1 and store 2 packets altogether. Packets arrive to the node according to a Poisson process. Serving a packet involves two independent sequentially performed tasks: the ERROR CHECK and the packet TRANSMISSION to the output link. Each task requires an exponentially distributed time with an average of 30msec. Give, that we observe that the node is empty in 60% of the time, what is the average time spend in the node for one packet?

Solution: As always, we need to construct the state diagram is such a way that all transitions rates are guaranteed to be exponential. The selected state space:

S_0: Empty network node,
S_{11}: One packet under transmission,
S_{10}: One packet under error-check,
S_{20}: One packet under error-check and one buffered,
S_{21}: One packet under transmission and one buffered,

The state diagram is shown in Fig. 3. We can form the *global* balance equations parameterized by λ. Then we apply information that is given: $P_0 = 0.6$; This extra information enables the solution of the system of equations, and leads to the calculation of λ:

$$\lambda P_0 = \mu P_{11}$$
$$(\lambda + \mu) P_{11} = \mu P_{10}$$
$$(\lambda + \mu) P_{10} = \lambda P_0 + \mu P_{21}$$
$$\mu P_{21} = \lambda P_{11} + \mu P_{22}$$
$$P_{11} + P_{10} + P_{21} + P_{22} = 1 - P_0 = 0.4$$

Solution: $P_{10} \approx 0.1636, P_{11} \approx 0.1337, P_{20} \approx 0.0365, P_{21} \approx 0.0663, \lambda \approx 7.63$. For the calculation of the total average *system* time for a packet, we apply

Little's formula.

$$\overline{N} = \lambda_{eff} \cdot E[T_{sys}] \rightarrow E[T_{sys}] = \frac{\overline{N}}{\lambda_{eff}} = \frac{1 \cdot (P_{10} + P_{11}) + 2 \cdot (P_{21} + P_{22})}{\lambda_{eff}}.$$

We always apply the effective arrival rate at Little's formula, because the formula needs the actual average arrival rate at the system, excluding possible drops. Here, the effective rate is not equal to λ, since we have packet drops. However, since the arrival rate for this system does not change with time, the effective arrival rate is simply:

$$\lambda_{eff} = \lambda \cdot (P_0 + P_{10} + P_{11}).$$