Spectrum

Deterministic Signals with Finite Energy ($l^2$)

Energy Spectrum: \[ S_{xx}(f) = |X(f)|^2 = \left| \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi fn} \right|^2 \]

Deterministic Signals with Infinite Energy

DTFT of truncated signal: \[ X_N(f) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi fn} \]

Power spectral density: \[ P_{xx}(f) = \lim_{N \to \infty} \frac{1}{N} |X_N(f)|^2 \]

Stochastic Signals

Power spectral density:

\[ P_{xx}(f) = \lim_{N \to \infty} \mathbb{E} \left\{ \frac{1}{N} |X_N(f)|^2 \right\} = \sum_{k=-\infty}^{\infty} r_{xx}(k)e^{-j2\pi fk} \]

where \( r_{xx}(k) = \mathbb{E}\{x(n)x^*(n-k)\} \) is the covariance sequence of the stationary stochastic process.
**Spectrum of Filtered Signal**

\[ x(n) \xrightarrow{\text{LTI: } h(n)} y(n) \]

**LTI system:** \( y(n) = h(n) * x(n) \)

**Frequency response:** \( Y(f) = H(f)X(f) \)

**Energy spectral density:** \((l_2\text{-signals})\) \( S_{yy}(f) = |Y(f)|^2 = S_{xx}(f)|H(f)|^2 \)

**Power spectral density:** (deterministic signals)
\[
P_{yy}(f) = \lim_{N \to \infty} \frac{1}{N} |Y_N(f)|^2 = P_{xx}(f)|H(f)|^2
\]

**Power spectral density:** (stochastic signals)
\[
P_{yy}(f) = \mathbb{E} \lim_{N \to \infty} \left\{ \frac{1}{N} |Y_N(f)|^2 \right\} = P_{xx}(f)|H(f)|^2
\]

---

**Sampling**

\[ x_c(t) \xrightarrow{t = nT_s} x_d(n) \]

\[ X_c(F) \xrightarrow{\text{Filter}} X_d(f) \]

**Sampling frequency:** \( F_s = \frac{1}{T_s} \)

**Poisson’s summation formula (the sampling theorem):**
\[
X_d(f) = F_s \sum_{k=-\infty}^{\infty} X_c((f - k)F_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c\left(\frac{f - k}{T_s}\right)
\]
SAMPLING STOCHASTIC SIGNALS

Covariance sequence:

$$r_{x_d x_d}(k) = r_{x_c x_c}(k T_s)$$

Power Spectral Density:

$$P_{x_d x_d}(f) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} P_{x_c x_c} \left( \frac{f - k}{T_s} \right)$$

NON-PARAMETRIC SPECTRAL ESTIMATION

Infinite (true) stochastic process: $$x(n), n = \ldots, -2, -1, 0, 1, 2, \ldots$$

Available data: $$N$$ samples of a single realization: $$x_N(n), n = 0, 1, 2, \ldots, N - 1$$

Problem formulation: Estimate the power spectral density of $$x(n)$$, given $$x_N(n)$$
Stochastic signals

Power spectral density:

\[ P_{xx}(f) = \lim_{{N \to \infty}} E \left\{ \frac{1}{N} |X_N(f)|^2 \right\} = \cdots = \sum_{k=-\infty}^{\infty} r_{xx}(k)e^{-jfk} \]

Estimate: \( \hat{P}_{xx}(f) \)

i) directly: from \( |F\{x_N(n)\}|^2 \)

ii) indirectly: from \( F\{\hat{r}_{xx}(k)\} \)

PERIODOGRAM

Directly:

\[ \hat{P}_{xx}(f) = \frac{1}{N} |F\{x_N(n)\}|^2 = \frac{1}{N} \left| \sum_{n=0}^{N-1} x(n)e^{-j2\pi fn} \right|^2 \]

Indirectly:

\[ \hat{P}_{xx}(f) = F\{\hat{r}_{xx}(k)\} = \sum_{k=-N+1}^{N-1} \hat{r}_{xx}(k)e^{-j2\pi fk} \]

if

\[ \hat{r}_{xx}(k) = \begin{cases} \frac{1}{N} \sum_{n=0}^{N-k-1} x(n+k)x^*(n) & k = 0, 1, \ldots, N - 1 \\ \hat{r}_{xx}^*(-k) & k = -1, -2, \ldots, -N + 1 \end{cases} \]
PERIODOGRAM, PROPERTIES

Bias:  $E\{\hat{P}_{xx}(f)\} = P_{xx}(f) \ast |W_R(f)|^2 = P_{xx}(f) + O\left(\frac{1}{N}\right)$
biased but asymptotically unbiased

Variance:  $E\{(\hat{P}_{xx}(f) - P_{xx}(f))^2\} = P_{xx}^2(f) + O\left(\frac{1}{N}\right)$
does not go to zero! $\implies$ not a consistent estimate!

Cross variance (different frequencies, $f_1 \neq f_2$)
$E\{(\hat{P}_{xx}(f_1) - P_{xx}(f_1))(\hat{P}_{xx}(f_2) - P_{xx}(f_2))\} = O\left(\frac{1}{N}\right)$
estimates at different frequencies weakly correlated for large $N$
$\implies$ not a smooth estimator!

MODIFIED PERIODOGRAM

Window the data:

$\hat{P}^M_{xx}(f) = \frac{1}{NU} |\mathcal{F}\{w(n)x_N(n)\}|^2 = \frac{1}{NU} \left| \sum_{n=0}^{N-1} w(n)x(n)e^{-j2\pi fn} \right|^2$

Normalization:  $U = \frac{1}{N} \sum_{n=0}^{N-1} |w(n)|^2$

Properties:  Changes the $O\left(\frac{1}{N}\right)$ term of the bias and variance.

Choice of window $w(n)$:  Trade-off between resolution and leakage, see Table 8.2 in Hayes.
**Windowing Effects**

*Side-lobes* cause *leakage*, i.e. energy appears outside the main lobe. The width of *main lobe* determines the *resolution* capabilities.

<table>
<thead>
<tr>
<th>Time domain</th>
<th>Frequency domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>long window</td>
<td>⇐⇒ narrow main-lobe</td>
</tr>
<tr>
<td>short window</td>
<td>⇐⇒ wide main-lobe</td>
</tr>
<tr>
<td>“sharp” edges</td>
<td>⇐⇒ large side-lobes</td>
</tr>
</tbody>
</table>

Different windows: trade-off between *resolution* and *leakage*.

**Resolution Limits**

Resolution if $|f_2 - f_1| < W!$

$W$ — “3dB bandwidth”
**BARTLETT METHOD**

Idea: Segment and average the data to decrease the variance.

**Number of segments:** $K$

**Length of each segment:** $L$

**Total data length:** $N = LK$

Segment $k$: $x_k(n) = x(kL + n), \ n = 0, \ldots, L - 1, \ k = 0, \ldots, K - 1$

$$\hat{P}^B_{xx}(f) = \frac{1}{K} \sum_{k=0}^{K-1} \frac{1}{L} |\mathcal{F}\{x_k(n)\}|^2$$

**BARTLETT METHOD, PROPERTIES**

Compared to the periodogram:

**Variance:** decrease by factor $K$. 😊

**Bias:** $\propto \frac{1}{L}$, — increase by factor $K$. 😞

**Resolution:** decrease by factor $K$. 😞
WELCH METHOD

Idea: Allow overlapping segments and window the data.

Number of segments: $K$

Length of each segment: $L, LK > N$.

Step between segment starts: $D$

Segment $k$: $x_k(n) = x(kD + n), n = 0, \ldots, L - 1, k = 0, \ldots, K - 1$

Temporal window: $w(n), n = 0, \ldots, L - 1$

\[
\hat{P}^{W}_{xx}(f) = \frac{1}{K} \sum_{k=0}^{K-1} \frac{1}{LU} \left| \mathcal{F} \{ w(n)x_k(n) \} \right|^2
\]

Modified Periodogram of segment $k$

Normalization: $U = \frac{1}{L} \sum_{n=0}^{L-1} |w(n)|^2$

WELCH METHOD, PROPERTIES

Window choice: Ordinary trade-off between resolution and leakage. Gives increased smoothness (averaging in the frequency domain).

Variance: Slightly lower than for Bartlett when using 50% overlap between segments ($L = 2D$).
**BLACKMAN-TUKEY METHOD**

**Covariance sequence estimate:** \( \hat{r}_{xx}(k) \)

**Idea:** \( \hat{r}_{xx}(k) \) is less reliable for large \( k \). Window the correlation sequence to put more emphasis on the most reliable values.

\[
\hat{P}_{XX}^{BT}(f) = \mathcal{F}\{w(k)\hat{r}_{xx}(k)\} = \sum_{k=-M+1}^{M-1} w(k)\hat{r}_{xx}(k)e^{-j2\pi fk}
\]

**Correlation window:**

\[
\begin{align*}
  w(k), n &= -M + 1, \ldots, -1, 0, 1, \ldots, M - 1 \\
  w(k) &= w(-k) \\
  w(0) &= 1 \iff \int_{-1/2}^{1/2} W(f) df = 1
\end{align*}
\]

**Effective window length:** \( M \leq N \).

**BLACKMAN-TUKEY METHOD, PROPERTIES**

**Bias:**

\[
E\{\hat{P}_{XX}^{BT}(f)\} = E\{\hat{P}_{XX}(f)\} \ast W(f) = P_{xx}(f) \ast W_{\text{triangle}}(f) \ast W(f)
\]

**Variance:** Decreases by approximately \( M/N \), compared to the Periodogram.

Typically \( M \ll N \).

**Smoothness:** Windowing creates smooth estimate.
**USING THE FFT**

**In practice:** Use the FFT to calculate \( \hat{P}_{xx}(f) \) in all the methods.

**Frequency axis:** \( \hat{P}_{xx}(k) = \hat{P}_{xx}(f) \mid f = \frac{k}{M} \), where \( M \) is the length of the DFT.

**Zero padding:** Use zero padding, \( M > N \), to get more points on the curve.