SPECTRUM

Deterministic Signals with Finite Energy (l_2)

Energy Spectrum: $S_{xx}(f) = |X(f)|^2 = \left| \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi fn} \right|^2$

Deterministic Signals with Infinite Energy

DTFT of truncated signal: $X_N(f) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi fn}$

Power spectral density: $P_{xx}(f) = \lim_{N\to\infty} \frac{1}{N} |X_N(f)|^2$

SPECTRUM

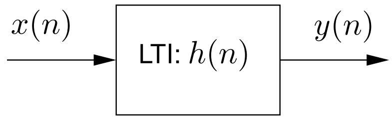
Stochastic Signals

Power spectral density:

$$P_{xx}(f) = \lim_{N\to\infty} \mathsf{E}\left\{ \frac{1}{N} |X_N(f)|^2 \right\} = \cdots = \sum_{k=-\infty}^{\infty} r_{xx}(k) e^{-j2\pi fk}$$

where $r_{xx}(k) = \mathsf{E}\{x(n)x^*(n-k)\}$ is the covariance sequence of the stationary stochastic process.

Spectrum of Filtered Signal



LTI system: $y(n) = h(n) * x(n)$

Frequency response: $Y(f) = H(f)X(f)$

Energy spectral density: (l_2-signals) $S_{yy}(f) = |Y(f)|^2 = S_{xx}(f)|H(f)|^2$

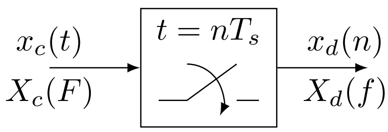
Power spectral density: (deterministic signals)

$$P_{yy}(f) = \lim_{N \to \infty} \frac{1}{N} |Y_N(f)|^2 = P_{xx}(f)|H(f)|^2$$

Power spectral density: (stochastic signals)

$$P_{yy}(f) = \mathsf{E} \lim_{N \to \infty} \left\{ \frac{1}{N} |Y_N(f)|^2 \right\} = P_{xx}(f)|H(f)|^2$$

Sampling



Sampling frequency: $F_s = \frac{1}{T_s}$

Poisson's summation formula (the sampling theorem):

$$X_d(f) = F_s \sum_{k=-\infty}^{\infty} X_c((f-k)F_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c\left(\frac{f-k}{T_s}\right)$$

SAMPLING STOCHASTIC SIGNALS

Covariance sequence:

$$r_{x_d x_d}(k) = r_{x_c x_c}(kT_s)$$

Power Spectral Density:

$$P_{x_d x_d}(f) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} P_{x_c x_c}\left(\frac{f-k}{T_s}\right)$$

NON-PARAMETRIC SPECTRAL ESTIMATION

Infinite (true) stochastic process: $x(n), n = \ldots, -2, -1, 0, 1, 2, \ldots$

Available data: N samples of a single realization: $x_N(n)$,
$n = 0, 1, 2, \ldots, N-1$

Problem formulation: Estimate the power spectral density of $x(n)$, given $x_N(n)$

Non-parametric Spectral Estimation

Stochastic signals

Power spectral density:

$$P_{xx}(f) = \lim_{N\to\infty} \mathsf{E}\left\{\frac{1}{N}\left|X_N(f)\right|^2\right\} = \cdots = \sum_{k=-\infty}^{\infty} r_{xx}(k)e^{-jfk}$$

Estimate: $\hat{P}_{xx}(f)$

i) **directly**: from $\left|\mathcal{F}\{x_N(n)\}\right|^2$

ii) **indirectly**: from $\mathcal{F}\{\hat{r}_{xx}(k)\}$

Periodogram

Directly:

$$\hat{P}_{xx}(f) = \frac{1}{N}\left|\mathcal{F}\{x_N(n)\}\right|^2 = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)e^{-j2\pi fn}\right|^2$$

Indirectly:

$$\hat{P}_{xx}(f) = \mathcal{F}\{\hat{r}_{xx}(k)\} = \sum_{k=-N+1}^{N-1} \hat{r}_{xx}(k)e^{-j2\pi fk}$$

if

$$\hat{r}_{xx}(k) = \begin{cases} \frac{1}{N}\sum_{n=0}^{N-k-1} x(n+k)x^*(n) & k = 0, 1, \ldots, N-1 \\ \hat{r}^*_{xx}(-k) & k = -1, -2, \ldots, -N+1 \end{cases}$$

PERIODOGRAM, PROPERTIES

Bias: $\mathrm{E}\{\hat{P}_{xx}(f)\} = P_{xx}(f) * |W_R(f)|^2 = P_{xx}(f) + \mathcal{O}(\frac{1}{N})$

biased but asymptotically unbiased

Variance: $\mathrm{E}\{(\hat{P}_{xx}(f) - P_{xx}(f))^2\} = P_{xx}^2(f) + \mathcal{O}(\frac{1}{N})$

does not go to zero! $\Longrightarrow$ not a consistent estimate!

Cross variance (different frequencies, $f_1 \neq f_2$)

$\mathrm{E}\{(\hat{P}_{xx}(f_1) - P_{xx}(f_1))(\hat{P}_{xx}(f_2) - P_{xx}(f_2))\} = \mathcal{O}(\frac{1}{N})$

estimates at different frequencies weakly correlated for large N

$\Longrightarrow$ not a smooth estimator!

MODIFIED PERIODOGRAM

Window the data:

$$\hat{P}_{xx}^M(f) = \frac{1}{NU}|\mathcal{F}\{w(n)x_N(n)\}|^2 = \frac{1}{NU}\left|\sum_{n=0}^{N-1} w(n)x(n)e^{-j2\pi fn}\right|^2$$

Normalization: $U = \frac{1}{N}\sum_{n=0}^{N-1}|w(n)|^2$

Properties: Changes the $\mathcal{O}(\frac{1}{N})$ term of the bias and variance.

Choice of window $w(n)$**:** Trade-off between resolution and leakage, see Table 8.2 in Hayes.

Windowing effects

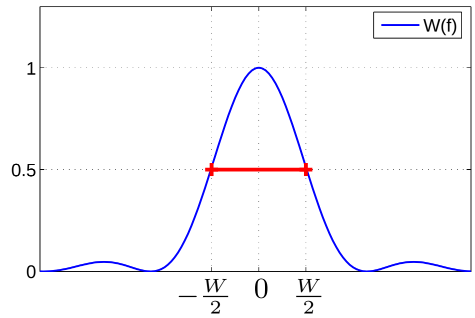
Side-lobes cause **leakage**, i.e. energy appears outside the main lobe.

The width of **main lobe** determines the **resolution** capabilities.

Time domain		Frequency domain
long window	$\Longleftrightarrow$	narrow main-lobe
short window	$\Longleftrightarrow$	wide main-lobe
"sharp" edges	$\Longleftrightarrow$	large side-lobes

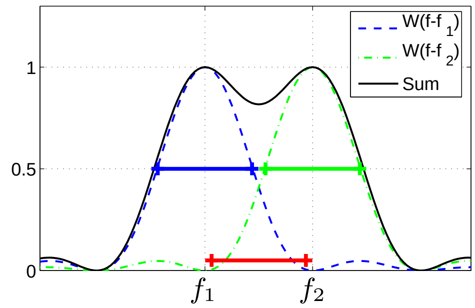
Different windows: trade-off between **resolution** and **leakage**.

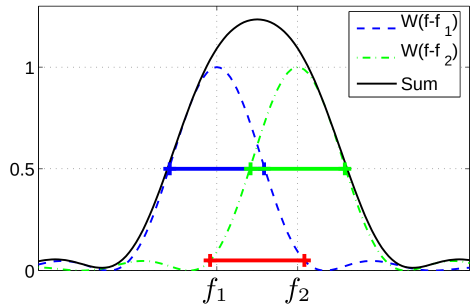
Resolution Limits



Resolution if $|f_2 - f_1| < W$!

W — "3dB bandwidth"





Bartlett Method

Idea: Segment and average the data to decrease the variance.

Number of segments: K

Length of each segment: L

Total data length: $N = LK$

Segment k**:** $x_k(n) = x(kL+n), n = 0, \ldots, L-1, k = 0, \ldots, K-1$

$$\hat{P}_{xx}^{B}(f) = \frac{1}{K}\sum_{k=0}^{K-1} \underbrace{\frac{1}{L}\left|\mathcal{F}\left\{x_k(n)\right\}\right|^2}_{\text{Periodogram of segment } k}$$

Bartlett Method, Properties

Compared to the periodogram:

Variance: decrease by factor K. ☺

Bias: $\propto \frac{1}{L}$, — increase by factor K. ☹

Resolution: decrease by factor K. ☹

Welch method

Idea: Allow overlapping segments and window the data.

Number of segments: K

Length of each segment: L, $LK > N$.

Step between segment starts: D

Segment k**:** $x_k(n) = x(kD + n), n = 0, \ldots, L-1, k = 0, \ldots, K-1$

Temporal window: $w(n), n = 0, \ldots, L-1$

$$\hat{P}_{xx}^{W}(f) = \frac{1}{K} \sum_{k=0}^{K-1} \underbrace{\frac{1}{LU} \left| \mathcal{F}\left\{ w(n) x_k(n) \right\} \right|^2}_{\text{Modified Periodogram of segment } k}$$

Normalization: $U = \dfrac{1}{L} \sum_{n=0}^{L-1} |w(n)|^2$

Welch method, properties

Window choice: Ordinary trade-off between resolution and leakage. Gives increased smoothness (averaging in the frequency domain).

Variance: Slightly lower than for Bartlett when using 50% overlap between segments ($L = 2D$).

BLACKMAN-TUKEY METHOD

Covariance sequence estimate: $\hat{r}_{xx}(k)$

Idea: $\hat{r}_{xx}(k)$ is less reliable for large k. Window the **correlation sequence** to put more emphasis on the most reliable values.

$$\hat{P}_{xx}^{BT}(f) = \mathcal{F}\{w(k)\hat{r}_{xx}(k)\} = \sum_{k=-M+1}^{M-1} w(k)\hat{r}_{xx}(k)e^{-j2\pi fk}$$

KTH Electrical Engineering

Correlation window:

$$w(k), n = -M+1, \ldots, -1, 0, 1, \ldots, M-1$$

$$w(k) = w(-k)$$

$$w(0) = 1 \iff \int_{-1/2}^{1/2} W(f)\, df = 1$$

Effective window length: $M \leq N$.

BLACKMAN-TUKEY METHOD, PROPERTIES

KTH Electrical Engineering

Bias: $\mathrm{E}\{\hat{P}_{xx}^{BT}(f)\} = \mathrm{E}\{\hat{P}_{xx}(f)\} * W(f) = P_{xx}(f) * W_{\text{triangle}}(f) * W(f)$

Variance: Decreases by approximately M/N, compared to the Periodogram. Typically $M \ll N$.

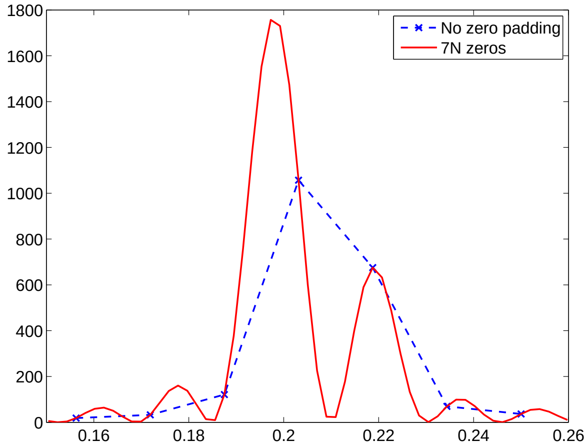
Smoothness: Windowing creates smooth estimate.

Using the FFT

In practice: Use the FFT to calculate $\hat{P}_{xx}(f)$ in all the methods.

Frequency axis: $\hat{P}_{xx}(k) = \hat{P}_{xx}(f)\big|_{f=\frac{k}{M}}$, where M is the length of the DFT.

Zero padding: Use zero padding, $M > N$, to get more points on the curve.



- - - Without zero padding

— With zero padding, $M = 8N$