EG2080 Monte Carlo Methods in Engineering

Theme 1

MONTE CARLO SIMULATION

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MONTE CARLO METHODS

- A class of methods used to solve mathematical problems by studying random samples.
- The name—which refers to the famous casino in Monaco—was coined in the beginning of the 20th century.
- Monte Carlo methods can be applied both to stochastic and deterministic problems.

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BUFFON’S NEEDLE

- In 1777, the French scientist Georges-Louis Leclerc, Comte de Buffon, described a method to calculate the value of π using a wooden floor and a needle.
- Toss the needle n times. Let x_i, $i = 1, \ldots, n$ be equal to 0 if the needle does not cross the line between two strips, and 1 if it does cross a line. Then

$$\pi = \frac{2l \cdot n}{d \cdot \sum_{i=1}^{n} x_i}$$

l = needle length,
d = distance between the lines ($d > l$).

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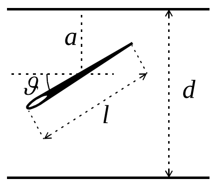
BUFFON’S NEEDLE

- The position of the needle can be described using two parameters:



a = least distance from the needle centre to one of the parallell lines ($0 \le a \le d/2$)

ϑ = least angle between the needle direction and the parallell lines ($0 \le \vartheta \le \pi/2$)

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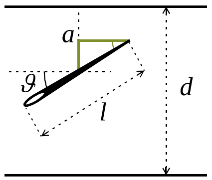
BUFFON'S NEEDLE

- The needle will cross a line if its projection on a line perpendicular to the parallel lines is larger than the distance to the closest parallel line, i.e., if



$$\frac{l}{2}\sin\vartheta \geq a.$$

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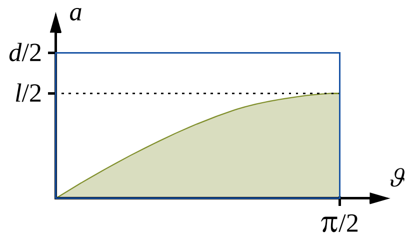
BUFFON'S NEEDLE



- The probability of the needle crossing a line is given by the area of the green surface divided by the area of the blue rectangle.

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BUFFON'S NEEDLE

- The probability can then be expressed as

$$P(X=1) = \frac{\int_0^{\pi/2} \frac{l}{2}\sin\vartheta}{\frac{d}{2}\cdot\frac{\pi}{2}} = \frac{2l}{d\cdot\pi}\int_0^{\pi/2}\sin\vartheta = \frac{2l}{d\cdot\pi}.$$

- If this probability is estimated by $\frac{1}{n}\sum_{i=1}^{n} x_i$ then we get

$$\frac{1}{n}\sum_{i=1}^{n} x_i \approx \frac{2l}{d\cdot\pi} \Rightarrow \pi \approx \frac{2l\cdot n}{d\cdot\sum_{i=1}^{n} x_i}. \blacksquare$$

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WHAT IS A SIMULATION?

- The word "simulation" is derived from Latin *simulare*, which means "to make like".
- A simulation is an attempt to imitate natural or technical systems.
- There are many reasons to simulate systems:
  - Understanding of the function of a system.
  - Predictions of the behaviour of a system.
  - Testing of a system.
  - Training for system users.

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SIMULATION METHODS

- Physical simulation. Study a copy of the original system.
  The copy usually is smaller and less expensive than the real system.
- Computer simulation. Study a mathematical model of the original system.
- Interactive simulation. Study a system (either physical or a computer simulation) and its human operators.

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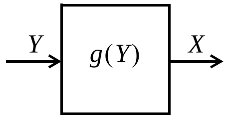
THE SIMULATION PROBLEM - Inputs



- The inputs are random variables with *known* probability distributions.
- For convenience, we collect all input variables in a vector, Y.

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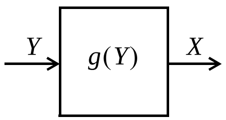
THE SIMULATION PROBLEM - Model



- The model is represented by the mathematical function, $g(Y)$.
- Notice that the random behaviour of the system is captured by the inputs, i.e., the model is deterministic!
  Hence, if $x_1 = g(y_1)$, $x_2 = g(y_2)$ and $y_1 = y_2$ then $x_1 = x_2$.

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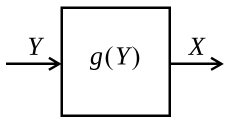
THE SIMULATION PROBLEM - Outputs



- The outputs are random variables with *unknown* probability distributions.
- For convenience, we collect all output variables in a vector, X.

The objective of the simulation is to study the probability distribution of X!

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MOTIVATION FOR MONTE CARLO SIMULATION

Assume that we want to calculate the expectation value, $E[X]$, of the system $X = g(Y)$.

According to the definition of expectation value we get the following expression:

$$E[X] = E[g(Y)] = \int_{\mathcal{Y}} f_Y(y)g(y)dy.$$

What reasons are there to solve this problem using Monte Carlo methods rather than analytical methods?

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MOTIVATION FOR MONTE CARLO SIMULATION

- Complexity. The model $g(y)$ may not be an explicit function.

  Example: The outputs, x, are given by the solution to an optimisation problem, where the inputs y appear as parameters, i.e.,

  $$\min_x z(x, y), \text{ subject to } h(x, y) = 0.$$

- Problem size. The model may have too many inputs or outputs.

  Example: 10 inputs $\Rightarrow$ integrate over 10 dimensions!

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EXAMPLE 1 - The Ice-cream Company

- The business idea of the Ice-cream Company (ICC) is to sell fresh-made ice-cream and sherbet to restaurants.
- Restaurants place their orders the day before delivery.
- The ice-cream is manufactured in the morning and delivered to the restaurants in the evening.

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EXAMPLE 1 - The Ice-cream Company

- Fresh ingredients are delivered in the morning. There is a certain risk that the delivery fails due to shortage of supply, transport problems, etc.
- The other ingredients are purchased on a weekly basis, and ICC have sufficient storage capacity to be certain that the ingredients are always available.

EXAMPLE 1 - The Ice-cream Company

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- The cost of the ingredients (in €/litre) is the same for all flavours.
- Each ice-cream machine has a certain maximal capacity (in litres) and operation cost (in €).*
- The restaurants pay the same price (in €/litre) is the same for all flavours.

* The operation cost does not depend on the amount of ice-cream prepared in the machine.

EXAMPLE 1 - The Ice-cream Company

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- The company always tries to deliver ice-cream if the ingredients are available, even if the manufacturing cost is higher than the price paid by the restaurants.
- Whenever a flavour cannot be delivered the company tries to replace it with another flavour.
- The replacement flavour is sold for half price.

EXAMPLE 1 - The Ice-cream Company

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Simulation objective

Determine the expected net income and the risk of not being able to deliver the ordered amount of ice-cream.

General

$\mathcal{F}$ = set of ice-cream flavours = $1, \ldots, F$

$\mathcal{I}$ = set of ingredients = {Milk+Cream, Eggs, Other}

$\mathcal{M}$ = set of ice-cream machines = $1, \ldots, M$

EXAMPLE 1 - The Ice-cream Company

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Inputs

A_i = ingredient i available

D_f = amount of flavour f to be delivered

Outputs

NI = net income

MD = missed ice-cream delivery

EXAMPLE 1 - The Ice-cream Company

Outputs (cont.)

$C_{m,f}$ = loading of machine m with flavour f

P_f = production of flavour f according to order

R_f = production of flavour f to replace another flavour

U = total amount of undelivered ice-cream

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EXAMPLE 1 - The Ice-cream Company

Model

Parameters:

λ = price of sold ice-cream

π_R, π_U = penalty factors for replacing a flavour with another and for undelivered ice-cream respectively

μ = ingredient cost of ice-cream

c_m = operation cost of machine m

γ_m = maximal capacity of machine m

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EXAMPLE 1 - The Ice-cream Company

Model (cont.)

$\alpha_{f,i}$ = critical ingredient index

The critical ingredient index shows which ingredients are necessary to make flavour f. The index values are chosen so that

$$\sum_{i \in I} \alpha_{f,i} = 1, \forall f \in \mathcal{F}$$

and $\alpha_{f,i} > 0$ if ingredient i is necessary to make ice-cream flavour f.

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EXAMPLE 1 - The Ice-cream Company

Model (cont.)

Planning problem—objective function:

$$\text{minimise} \quad \sum_{f \in \mathcal{F}} \pi_R R_f + \pi_U U + \sum_{m \in M} \sum_{f \in \mathcal{F}} c_m C_{m,f}$$

Minimise penalty costs of replacing flavours and penalty cost of undelivered ice-cream plus operation cost of machines.

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EXAMPLE 1 - The Ice-cream Company

Model (cont.)

Planning problem—constraints:

$$C_{m,f} \leq \sum_{i \in I} \alpha_{f,i} A_i \qquad m \in \mathcal{M}, f \in \mathcal{F}$$

The machines can only make an ice-cream flavour if all critical ingredients are available.

$$\sum_{f \in \mathcal{F}} C_{m,f} \leq 1, \qquad m \in \mathcal{M}$$

Each machine can only make one flavour.

EXAMPLE 1 - The Ice-cream Company

Model (cont.)

$$P_f + R_f \leq \sum_{m \in \mathcal{M}} \gamma_m C_{m,f}, \qquad f \in \mathcal{F}$$

The total production of an ice-cream flavour can not exceed the capacity of the machines making that flavour.

$$\sum_{f \in \mathcal{F}} (P_f + R_f) = \sum_{f \in \mathcal{F}} D_f - U$$

The total production of ice-cream should equal the total demand minus undelivered ice-cream.

EXAMPLE 1 - The Ice-cream Company

Model (cont.)

Planning problem—variable limits:

$$C_{m,f} \in \{0, 1\} \qquad m \in \mathcal{M}, f \in \mathcal{F}$$

$$0 \leq P_f \qquad f \in \mathcal{F}$$

$$0 \leq R_f \qquad f \in \mathcal{F}$$

$$0 \leq U$$

EXAMPLE 1 - The Ice-cream Company

Model (cont.)

Additional calculations:

$$NI = \sum_{f \in \mathcal{F}} \left((\lambda - \mu) P_f + \left(\frac{\lambda}{2} - \mu \right) R_f \right) - \sum_{m \in M} \sum_{f \in \mathcal{F}} c_m C_{m,f}$$

The net income is the income of sold ice-cream minus the ingredient and operation costs.

EXAMPLE 1 -
The Ice-cream Company

Model (cont.)

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$$MD = \begin{cases} 0 & \text{if } U = 0 \\ 1 & \text{if } U > 0 \end{cases}$$

A binary output is necessary to determine the risk of missing a delivery.