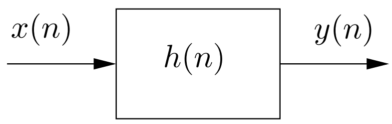
Model Based Signal Processing

Model of a system



Common model structures:

Auto-regressive, AR(N_a) $y(n) + a_1 y(n-1) + \ldots + a_{N_a} y(n - N_a) = u(n)$

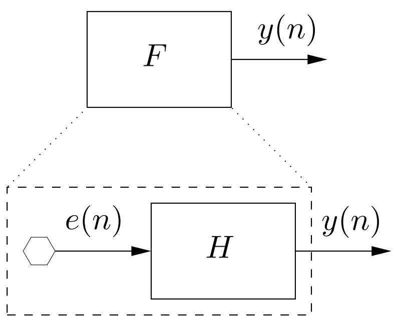
Moving Average, MA(N_b)

$$y(n) = b_0 u(n) + b_1 u(n-1) + \ldots + b_{N_b} u(n - N_b)$$

Auto-Regressive Moving Average, ARMA(N_a,N_b)

$$y(n) + a_1 y(n-1) + \ldots + a_{N_a} y(n - N_a) = b_0 u(n) + \ldots + b_{N_b} u(n - N_b)$$

Model of a Signal



$e(n)$**:** White noise, $r_{ee}(k) = \sigma_e^2 \delta(k)$

H**:** AR, MA, ARMA or other model

PARAMETER ESTIMATION PRINCIPLES

(1) "Method of Moments":

- Estimate correlation functions.
- Use theoretical relationships parameters $\Longleftrightarrow$ correlations.

(2) "Prediction Error Methods":

- Design a predictor from the model.
- Find parameters that minimize the prediction error.

ARMA SYSTEM

(If both $x(n)$ and $y(n)$ can be measured)

Correlation functions in theory:

$$r_{yy}(m) + a_1 r_{yy}(m-1) + \cdots + a_{N_a} r_{yy}(m - N_a)$$
$$= b_0 r_{uy}(m) + \cdots + b_1 r_{uy}(m-1) + b_{N_b} r_{uy}(m - N_b)$$

Method of moments:

- Estimate $\hat{r}_{yy}(m)$ and $\hat{r}_{uy}(m)$ for different values of m.
- Replace r with $\hat{r}$ in the theoretical expression.
- Use $N_a + N_b + 1$ different values of m.
- Solve for the parameters a_k and b_k.

ESTIMATION OF CORRELATION FUNCTIONS

In theory: $r_{uy}(k) = \mathrm{E}\{u(n)y^*(n-k)\}$

Estimate: $$\hat{r}_{uy}(k) = \begin{cases} \frac{1}{L}\sum_{n=k}^{N-1} u(n)y^*(n-k) & k \geq 0 \\ \frac{1}{L}\sum_{n=0}^{N-1+k} u(n)y^*(n-k) & k < 0 \end{cases}$$

$L = N - |k|$ gives **unbiased** estimates.

$L = N$ gives **biased** estimates, but more stable results for most algorithms.

AR(N) SIGNAL MODEL

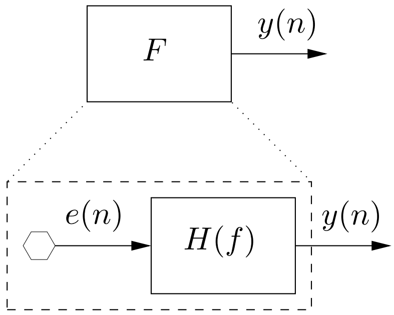
Method of moments $\Longrightarrow$ Yule-Walker equations:

$$\begin{bmatrix} r_{yy}(0) & r_{yy}^*(1) & \cdots & r_{yy}^*(N) \\ r_{yy}(1) & r_{yy}(0) & \cdots & r_{yy}^*(N-1) \\ \vdots & \vdots & \ddots & \vdots \\ r_{yy}(N) & r_{yy}(N-1) & \cdots & r_{yy}(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} \sigma_e^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

....

Alternative formulation: $$\begin{cases} \begin{bmatrix} r_{yy}(0) & \cdots & r_{yy}^*(N-1) \\ \vdots & \ddots & \vdots \\ r_{yy}(N-1) & \cdots & r_{yy}(0) \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} -r_{yy}(1) \\ \vdots \\ -r_{yy}(N) \end{bmatrix} \\ \sigma_e^2 = r_{yy}(0) + a_1 r_{yy}^*(1) + \cdots + a_N r_{yy}^*(N) \end{cases}$$

Model Based Spectral Estimation



- Estimate parameters: $\hat{H}$ and $\hat{\sigma}_e^2$.
- Study spectrum of the estimated model: $\hat{P}_{yy}(f) = \hat{\sigma}_e^2 |\hat{H}(f)|^2$

AR(N) signal model

AR(N) model: $y(n) + a_1 y(n-1) + \ldots + a_N y(n-N) = e(n)$,

$r_{ee}(k) = \sigma_e^2 \delta(k)$

Order N, 1-step ahead predictor:

$$\hat{y}(n) = \alpha_1 y(n-1) + \alpha_2 y(n-2) + \ldots \alpha_N y(n-N)$$

Equivalent if

- $\alpha_k = -a_k$
- MSE of the predictor = σ_e^2

Method of moments $\Longleftrightarrow$ MMSE prediction.

$\Longrightarrow$ Yule-Walker equations.

MODEL ORDER ESTIMATION

Problem: (over-fitting)

Best fit to one specific data set:
number of parameters = number of samples

Best fit to the signal in general and to other realizations: number of parameters $\ll$ number of samples

Solution 1:

- Split data into two halves: *Estimation* and *Validation data*
- Estimate parameters of $H(z)$ from the estimation data
- Estimate prediction error σ_e^2 from the validation data
- Use the order that minimizes the prediction error

Other solutions: Akaike (AIC),
Minimum Description Length (MDL), ...