Tutorial 8 – M/M/m systems with limited number of customers

Ioannis Glaropoulos

November 19, 2012

1 Exercise 6.6

There are K computers in an office, and a single repairman. Each computer breaks down after an exponentially distributed time with parameter α. The repair takes an exponentially distributed amount of time with parameter β. Only one computer is being repaired at a time, computers break down independently from the repair process, and repair times and lifetimes of the computers are independent.

1. *What is the probability that i computers are working at time t?*
2. *What is the average failure rate (i.e. the average number of computers that fail per time unit)?*
3. *What is the percentage of time a repairman is busy?*
4. *What is the percentage of time when all computers are out-of-order (broken)?*
5. *How many computers should we have if we would like to have K computers to work on average?*

Solution:

By reading the first question of this problem we already realize what the desired *State Space* could be. We try with the obvious selection:

$$\text{State} \quad S_i: \quad i \text{ computers working}, \quad i = 0, 1, 2, ..., K$$

The K computers in the system is the *population size* of our model. And it is, clearly, **finite**. The single repairman represents the *servers* in our model (1). Each computer needs an exponentially distributed time for repair, so the service rates in our model are exponential. Finally, each computer breaks-down after an exponentially distributed time (after its repair) so the arrival process in the system is, also, Markovian.

To summarize:

- A break-down of a computer is an "arrival"

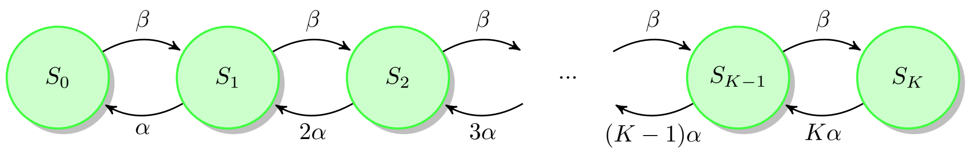

Figure 1: System diagram for exercise 6.6

- A repair of a computer is a "departure" or a "service"

Based on the above, we depict the diagram in Fig. 1. We denote $\gamma = \beta/\alpha$. We draw the Balance Equations:

$$\begin{aligned}
&\beta P_0 = \alpha P_1 \rightarrow P_1 = \tfrac{\beta}{\alpha} P_0 = \gamma P_0, \\
&\beta P_1 = 2\alpha P_2 \rightarrow P_2 = \tfrac{\beta}{2\alpha} P_1 = \tfrac{1}{2}\gamma P_1 = \tfrac{1}{2}\gamma^2 P_0, \\
&\beta P_1 = 3\alpha P_2 \rightarrow P_2 = \tfrac{\beta}{3\alpha} P_1 = \tfrac{1}{3}\gamma P_1 = \tfrac{1}{2\cdot 3}\gamma^3 P_0, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\beta P_{K-1} = K\alpha P_K \rightarrow P_K = \tfrac{\beta}{K\alpha} P_{K-1} = \ldots. = \tfrac{1}{K!}\gamma^K P_0.
\end{aligned} \tag{1}$$

From the above it is clear that the probability of an arbitrary state S_i is

$$P_i = \frac{\gamma^i}{i!} P_0, \quad i = 0, 1, 2, ..., K \tag{2}$$

From the above and the normalization equation we derive the P_0:

$$\sum_{j=1}^{K} P_j = 1 \rightarrow \sum_{j=1}^{K} \frac{1}{j!}\gamma^j P_0 = 1 \rightarrow P_0 = \frac{1}{\sum_{j=1}^{K} \frac{1}{j!}\gamma^j}. \tag{3}$$

P_0 gives the probability that all computers are out of order. In addition, the probability that i computers are working (in the steady-state) is given by:

$$P_i = \frac{\gamma^i}{i! \sum_{j=1}^{K} \frac{\gamma^j}{j!}}. \tag{4}$$

The repairman is busy in all states, except state S_K, where all computers are working properly. So the percentage of time the repairman is busy, will be:

$$P_{\text{rep. busy}} = 1 - P_K. \tag{5}$$

Next, we want to calculate the average failure rate ($\lambda_{\text{fail rate}}$). We notice that the failure rate **depends** on the state of the system. For example, the failure rate in state S_K is $K\alpha$, while, in case S_0 is it zero! We use the conditional expectation calculation to compute it:

$$\lambda_{\text{fail rate}} = \sum_{j=1}^{K} j \cdot \alpha P_j = \alpha \sum_{j=1}^{K} j \cdot P_j, \tag{6}$$

where P_j is given above. The average number of working computers will be calculated based on the state probability distribution, that is:

$$\overline{N} = \sum_{i=1}^{K} i \cdot P_i = \sum_{i=1}^{K} i \cdot \frac{\gamma^i}{i! \sum_{j=1}^{K} \frac{\gamma^j}{j!}} \tag{7}$$

We would like to have a system with K computers working (on average). Assume the required number of computers is M. Then M can be found as the lowest integer that satisfies the inequality:

$$\overline{N} \geq K \quad \rightarrow \quad \sum_{i=1}^{M} i \cdot \frac{\gamma^i}{i! \sum_{j=1}^{M} \frac{\gamma^j}{j!}} \geq K. \tag{8}$$

We can find the require M by trial-and-error.
Extra question: What is the probability that a broken computer can not be repaired immediately?
This probability can be found by taking the *ratio* of the rate of computer break-downs that can not be served immediately, over the total average rate of computer break-downs:

$$\lambda_W = \frac{\sum_{j=1}^{K-1} \alpha j P_j}{\sum_{j=1}^{K} \alpha j P_j} \tag{9}$$

2 Exercise 8.6

In a kitchen dormitory corridor there are two hobs for cooking and 3 places on the sofa. There are 8 students living in the corridor, each of them goes on average every $1/\alpha$ hours to the kitchen to cook (if he is not cooking already), the inter-arrival time is exponentially distributed. If on arrival the 2 hobs are occupied, the student looks for a place on the sofa. If the sofa is occupied as well, the student goes back to his room and tries again at a later time. Students spend an exponentially distributed amount of time cooking with mean $1/\beta$. $\alpha = 0.5$ hours, $\beta = 1$ hours.

1. *Draw the state transition diagram*
2. *Calculate the mean waiting time of the students*
3. *Calculate the ratio of time the kitchen is completely full, e.g. a student arriving has to go back to his room.*
4. *Calculate the probability that a student finds the kitchen completely full*
5. *Calculate the probability that a student has to wait more than 2 hours (supposing that he can sit down in the kitchen)*

Solution:

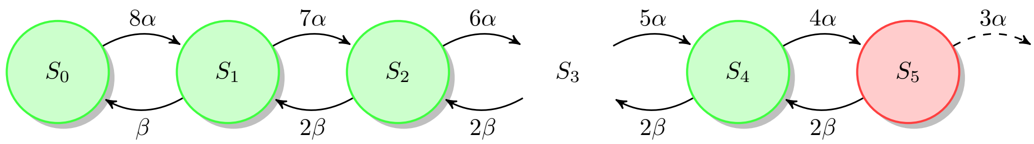

Figure 2: System diagram for exercise 8.6

This is a complex problem that discusses all important matters in the Markovian systems with finite population size. It is, perhaps, easy to choose the state space; it could, clearly, be the number of places occupied by students at some point in time, either in the kitchen counter (hobs) or the kitchen sofa:

$$\text{State Space:} \quad S_k, \quad k \text{ positions occupied}, \quad k = 0, 1, ..., 5. \tag{10}$$

The population of the system is the number of students (8). The number of servers is the number of hobs (2) and the number of queuing positions is the number of places on the sofa (3). The inter-arrival times between student arrivals at the kitchen is exponentially distributed and the rate depends on the remaining number of students that are still at their rooms (not already cooking or waiting at the sofa). The service times are the cooking times and are also exponentially distributed. Consequently, the system is Markovian, with Kendall notation: M/M/2/5/8. The system diagram is depicted in Fig. 2. We define:

$$\rho = \frac{\alpha}{\beta} = \frac{1}{2}.$$

Notice that is **not** the offered load at the system. We draw the balance equations:

$$\begin{aligned}
&8\alpha P_0 = \beta P_1 \to P_1 = 8\rho P_0, \\
&7\alpha P_1 = 2\beta P_2 \to P_2 = \tfrac{7}{2}\rho P_1 = 28\rho^2 P_0, \\
&6\alpha P_2 = 2\beta P_3 \to P_3 = 3\rho P_2 = 84\rho^3 P_0, \\
&5\alpha P_3 = 2\beta P_4 \to P_4 = \tfrac{5}{2}\rho P_3 = 210\rho^4 P_0, \\
&4\alpha P_4 = 2\beta P_5 \to P_5 = 2\rho P_2 = 420\rho^5 P_0
\end{aligned} \tag{11}$$

Applying the normalization equation:

$$\sum_{k=0}^{5} P_k = 1, \tag{12}$$

we calculate the values of the state probabilities.

Through the state probabilities we calculate, first, the percentage of time the kitchen is completely full, simply:

$$P_{\text{full}} = P_5. \tag{13}$$

Now, since the system has finite population, the arrival rates depend on the state, and, so the probability that a random student finds the system (kitchen) full is NOT equal to P_5. The **average student arrival rate** is calculated using the conditional expectation and taking into account the different arrival rates in each state:

$$\overline{\lambda} = \sum_{k=0}^{5} \lambda_k \cdot P_k, \tag{14}$$

where λ_k is the arrival rate at state S_k. Based on the diagram we obtain:

$$\overline{\lambda} = 8\alpha P_0 + 7\alpha P_1 + 6\alpha P_2 + 5\alpha P_3 + 4\alpha P_4 + 3\alpha P_3. \tag{15}$$

The **average student block rate** is the average rate of students that are blocked, i.e. find the kitchen completely full. Since they are only blocked at state S_5 we obtain:

$$\overline{\lambda}_{\text{block}} = 3\alpha P_5. \tag{16}$$

The ratio between the two average arrival rates gives the percentage of students that are blocked, or, equivalently, the probability that an arbitrary student is blocked (P_{blocked}):

$$P_{\text{blocked}} = \frac{\overline{\lambda}}{\overline{\lambda}_{\text{block}}} = \frac{3P_5}{8P_0 + 7P_1 + 6P_2 + 5P_3 + 4P_4 + 3P_3} \tag{17}$$

Clearly, the **effective** arrival rate of the system is the total average arrival rate minus the blocked arrival rate:

$$\lambda_{\text{eff}} = \overline{\lambda} - \overline{\lambda}_{\text{block}} = 8\alpha P_0 + 7\alpha P_1 + 6\alpha P_2 + 5\alpha P_3 + 4\alpha P_4. \tag{18}$$

We can find the mean waiting time of the student with the help of LITTLE's formula. First, we need to compute the average number of students in the kitchen:

$$\overline{N} = \sum_{i=0}^{5} kP_k = P_1 + 2P_2 + 3P_3 + 4P_4 + 5P_5. \tag{19}$$

Then, using LITTLE we can find the average SYSTEM time for a student:

$$E[T_{\text{system}}] = \frac{\overline{N}}{\lambda_{\text{eff}}}. \tag{20}$$

Then the average waiting time will be equal to the average system time minus the average service (cooking) time:

$$\overline{W} = E[T_{\text{system}}] - \frac{1}{\beta}. \tag{21}$$

There is another way to do this; that of considering the arrivals at each state and the waiting time a student experiences given the state of arrival. Here, we must reject the blocked arrivals cause they have no waiting time.

The last question is a bit challenging. We are asked to consider ONLY those students that are not rejected. So we must consider students that arrive at states S_k, $\quad k = 0, ...4$. Generally, we use the total probability theorem, and obtain:

$$\Pr\{W > 2h\} = \sum_{k=0}^{4} \Pr\{W > 2h|\text{Student sees state } S_k\} \cdot \Pr\{\text{Student sees state } S_k\} \tag{22}$$

1. If the student observes state S_0 or S_1 there is not waiting time. (Cooks immediately)
2. If the student observes state S_2 the student waits for an exponential amount of time with parameter 2β.
3. If the student observes state S_3 the student waits for a sum of two exponential amounts of time, each with parameter 2β.
4. If the student observes state S_4 the student waits for a sum of three exponential amounts of time, each with parameter 2β.

One option is to realize that the sum of exponential variables is an Erlang-distributed variable, and use the Erlang distribution formulas to calculate the $\Pr\{W > 2h|.....\}$.

There is however, a better option. We can realize that the times between departures are exponentially distributed variables with rate 2β, considering fully loaded kitchen counter. So, the number of departures within some time interval will be a Poisson random variable! This is the duality between the exponential and the Poisson distributions!

So, for example, if the student observes S_4 he will want for more than T=2h, if less than 3 departures occur during these two hours, and the number of these departures is Poisson $(2\beta \cdot T)$:

$$\begin{aligned}
\Pr\{W > T|\text{Student sees state } S_4\} &= \left(\tfrac{(2\beta T)^0}{0!} + \tfrac{(2\beta T)^1}{1!} + \tfrac{(2\beta T)^2}{2!}\right) e^{-2\beta T} \\
\Pr\{W > T|\text{Student sees state } S_3\} &= \left(\tfrac{(2\beta T)^0}{0!} + \tfrac{(2\beta T)^1}{1!}\right) e^{-2\beta T} \\
\Pr\{W > T|\text{Student sees state } S_2\} &= \left(\tfrac{(2\beta T)^0}{0!} +\right) e^{-2\beta T}
\end{aligned} \tag{23}$$

The last thing to calculate the probabilities that a random student observes a particular state. Using the same reasoning as in the calculation of the blocked students, we get:

$$\begin{aligned}
\Pr\{\text{Student sees state } S_2\} &= \tfrac{6\alpha P_2}{\lambda_{\text{eff}}} \\
\Pr\{\text{Student sees state } S_2\} &= \tfrac{5\alpha P_3}{\lambda_{\text{eff}}} \\
\Pr\{\text{Student sees state } S_2\} &= \tfrac{4\alpha P_4}{\lambda_{\text{eff}}}
\end{aligned} \tag{24}$$

Notice that we used λ_{eff} because we exclude those students that arrive at state S_5 and are blocked (because they have no waiting time).