EP2200 Queueing theory and teletraffic systems

Lecture 8

Semi-Markovian systems
The method of stages

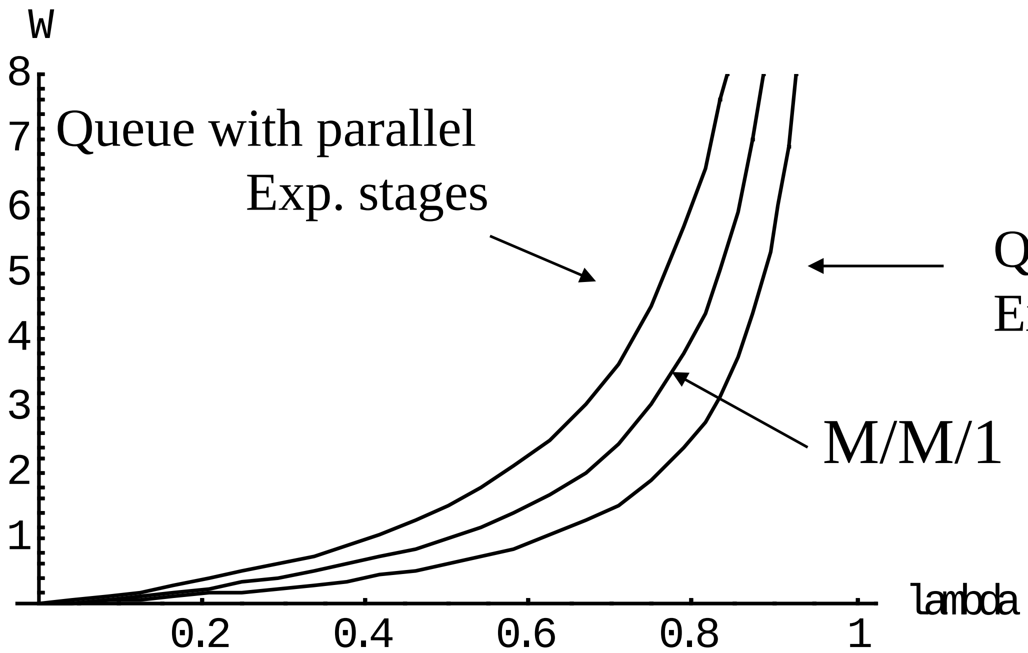
Viktoria Fodor
KTH EES/LCN

Semi-markovian system

- Advantages with M/M/*
  - The interarrival time and the service time distribution is memoryless
  - The state can be defined by the number of customers in the system
- Applicability for real systems
  - The arrival process is often Poisson (large number of potential customers)
  - The service process is often **not** memoryless
    - E.g., packet size distribution on the Internet, file size distribution
    - The future of the system depends on the elapsed service time
- Ways to handle the non-exponential service time – semi-markovian systems (semi-markov: MC to describe prossible state transitions, but the holding times are not exponential)
  1. Look at distributions consisting of several exponentially distributed stages in series or in parallel
  2. Describe the system only at specific points of time (e.g., end of service) – M/G/1

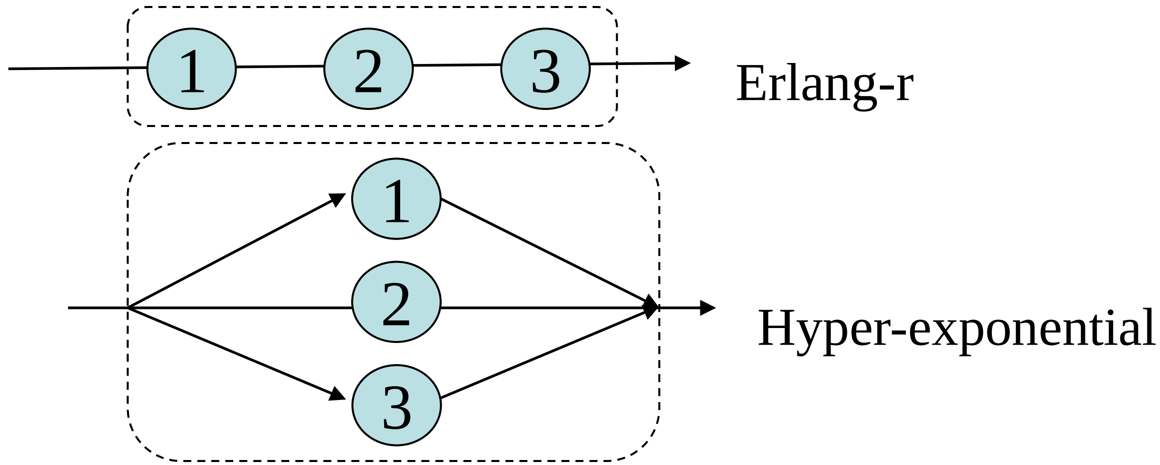
Semi-markovian systems – method of stages

- Use distributions that are composed of Exponential distributions

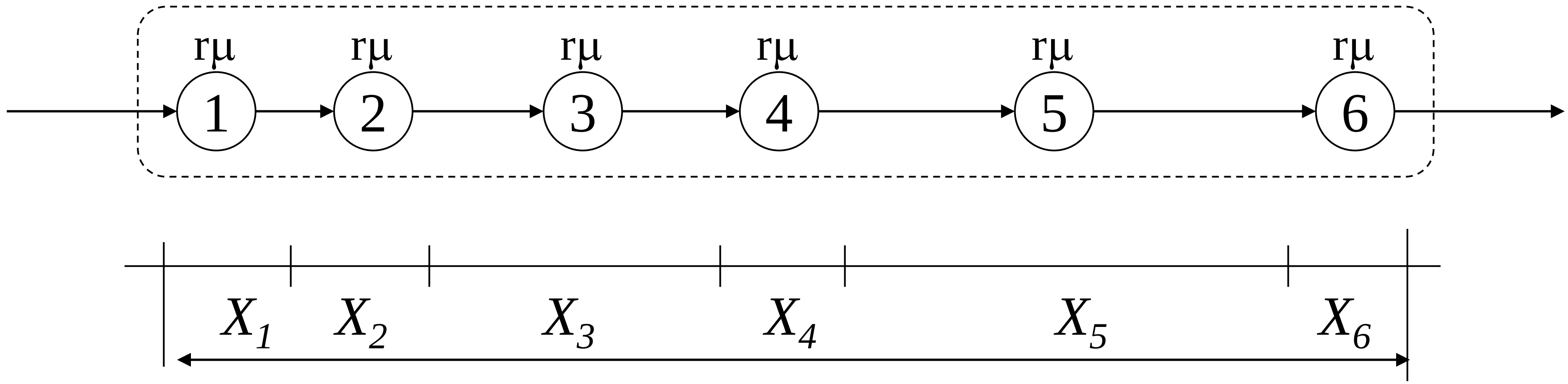


The method of stages

- Each service stage is Exponential
- Series of stages: the customer has to finish r service stages before the next customer can enter the server → Erlang-r service time distribution
- Parallel (or alternative) stages: the customer selects one server randomly, but only one customer can be in the service unit → Hyper-exponential service time distribution (linear combination of Exp. distributions)



Erlang-r server (E_r)



X is E_6 distributed, E[X]=r/(rμ)=1/μ

- Goal: service time with average $E[X] = \bar{x} = {}^{1}/_{\mu}$
- Since $E[X] = \sum E[X_i]$ if $X = \sum X_i$, we select:
  - X_i a stochastic variable with Exponential distribution $b(x_i) = r\mu e^{r\mu x_i}$
  - $X = \sum_{i=1}^{r} X_i$, X_i, X_j independent, identically distributed
  - *That is,* X is Erlang-r distributed

Erlang-r server (E_r)

- For each exponential stage:

$$b(x_i) = r\mu e^{r\mu x_i}$$

$$\left.\begin{aligned} E[X_i] &= \frac{1}{r\mu} \\ V[X_i] &= \left(\frac{1}{r\mu}\right)^2 \end{aligned}\right\} \quad C_{x_i}^2 \overset{\Delta}{=} \frac{V[X_i]}{E[X_i]^2} = 1$$

(coefficient of variation)

- For the service time:

$$X = X_1 + X_2 + \ldots X_r$$

$$b(x) = \frac{(r\mu)^r x^{r-1}}{(r-1)!} e^{-r\mu x} \quad (Erlang - r)$$

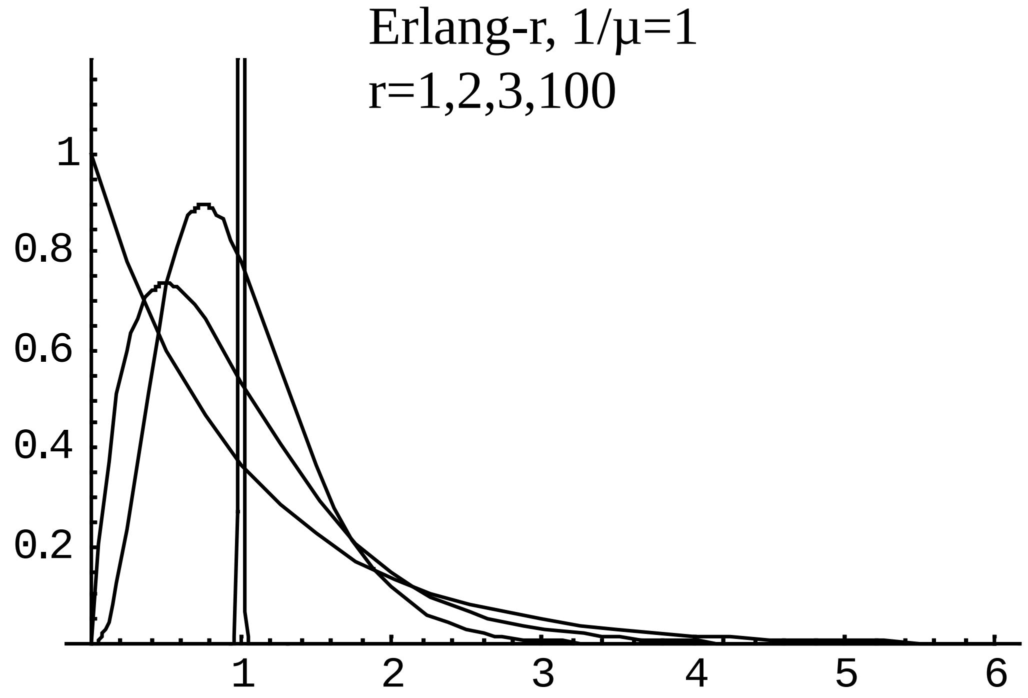
$$\left.\begin{aligned} E[X] &= rE[X_i] = \frac{1}{\mu} \\ V[X] &= rV[X_i] = \frac{1}{r\mu^2} \end{aligned}\right\} \quad C_x^2 = \frac{1}{r} \quad < 1$$

Erlang-r server (E_r)

$$X = X_1 + X_2 + \dots X_r$$

$$L(b(x)) = \left(\frac{r\mu}{s + r\mu}\right)^r, \quad b(x) = \frac{(r\mu)^r x^{r-1}}{(r-1)!} e^{-r\mu x}$$

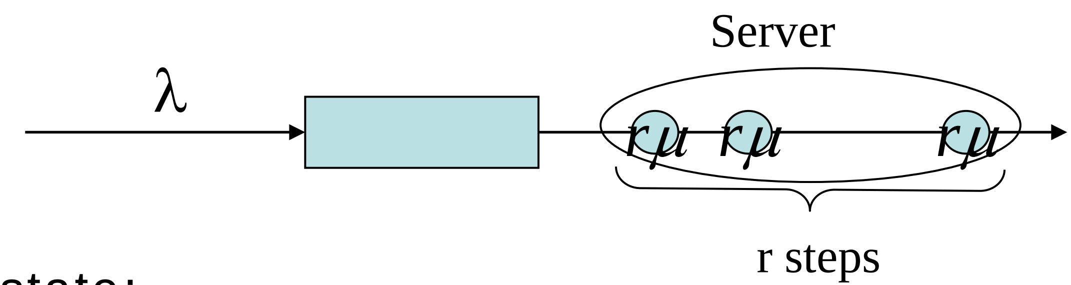
$$\left.\begin{array}{l} E[X] = rE[X_i] = \dfrac{1}{\mu} \\ V[X] = rV[X_i] = \dfrac{1}{r\mu^2} \end{array}\right\} \quad C_x^2 = \frac{1}{r} \quad < 1$$



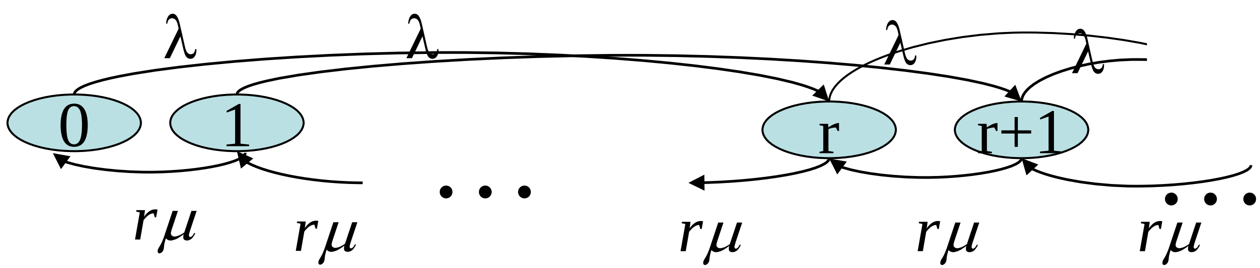
- As r→∞, V[X]→0, which means deterministic service time!

The M/E$_r$/1- queue

- If the system to be modeled has serial service or the service distribution has $C_x^2 < 1$ – approximate with Erlang-r

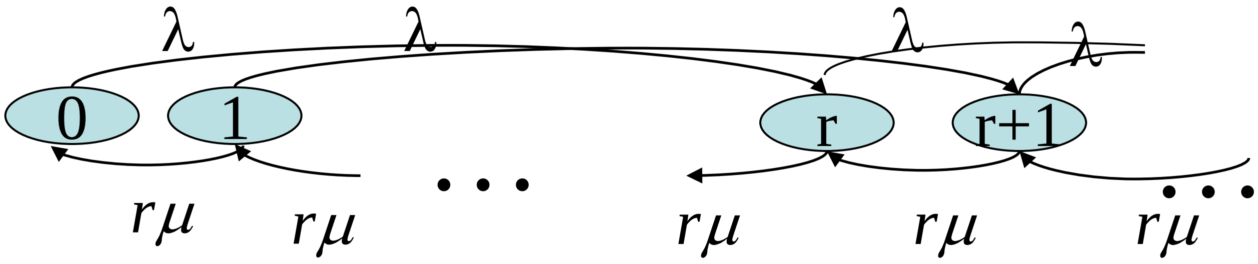


- System state:
  - {number of remaining service stages, number of customers}, or
  - number of remaining service stages + r*number of waiting customers
- The system can be modeled as a Markov chain



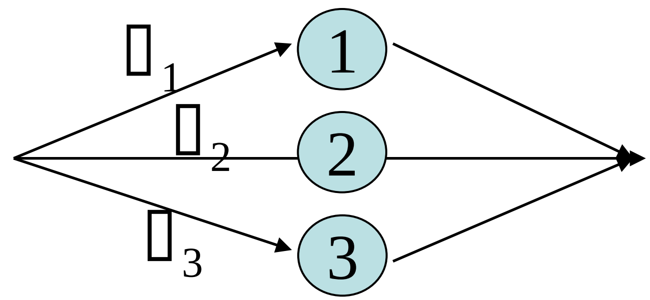
The M/E$_r$/1- queue

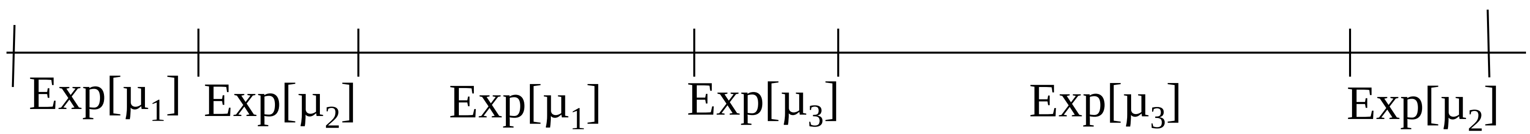
- System state:
  – number of remaining service stages + r*number of waiting customers
- Number of customers in the system in state i: $N_i = \lceil i/r \rceil$
- State probability distribution with z-transforms (Kleinrock p.127-128)
  – (not exam material)
- But, the followings hold:
  – PASTA
  – Little: $N_s = \lambda x = \lambda/\mu$ = Utilization
  – For r=1: M/M/1, for r=∞: M/D/1
  – You will have to be able to calculate state probabilities and performance measures for limited buffer systems (e.g., *M/E$_2$/1/3)*!
  – Average performance for M/E$_r$/1 with general forms of M/G/1



Hyper-exponential server (H_r)

- r exponential servers with different μ_i-s
- Server *i* is chosen with the probability α_i
  - E.g., different types of packets intermixed
  - service time distribution is the linear combination (mixture) of Exp distributions





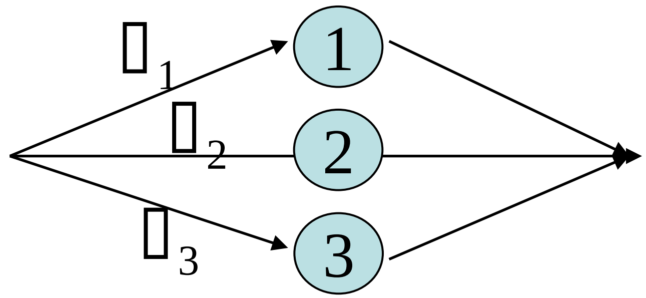
a possible sequence of service of 6 customers

$$b(x_i) = \mu_i e^{-\mu_i x}$$

$$b(x) = \alpha_1 \mu_1 e^{-\mu_1 x} + \ldots + \alpha_R \mu_R e^{-\mu_R x}, \quad \sum \alpha_i = 1$$

The hyper-exponential server (H_r)

- r exponential servers with different μ-s $B(x_i) = 1 - e^{-\mu_i x}$
- *Server i* is chosen with the probability α_i



$$b(x) = \alpha_1 \mu_1 e^{-\mu_1 x} + \ldots + \alpha_R \mu_R e^{-\mu_R x}, \quad \sum \alpha_i = 1$$

$$L(b(x)) = \sum_{i=1}^{r} \alpha_i \frac{\mu_i}{s + \mu_i}$$

$$\left.\begin{aligned} E[X] &= \sum_i \frac{\alpha_i}{\mu_i} \\ E[X^2] &= \sum_i \alpha_i \frac{2}{\mu_i^2} \\ V[X] &= E[X^2] - E[X]^2 \end{aligned}\right\} \quad C_{x_i}^2 = \frac{V[X_i]}{E[X_i]^2} = \frac{E[X_i^2] - E[X_i]^2}{E[X_i]^2} = \frac{E[X_i^2]}{E[X_i]^2} - 1$$

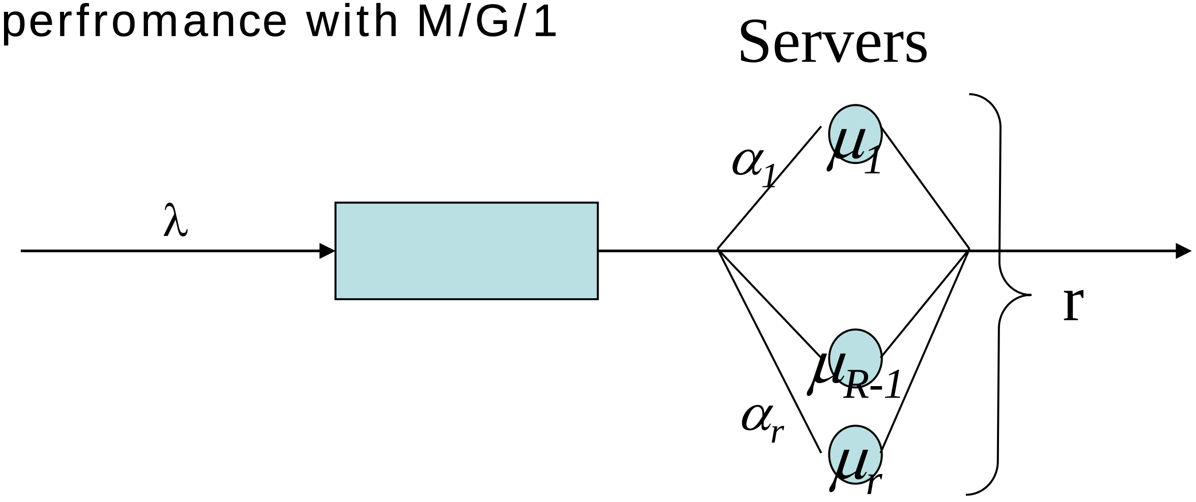
$$C_x^2 = \frac{E[X^2]}{E[X]^2} - 1 \quad \geq 1$$

- *For given coefficient of variation 2R-1* free parameters in total
  - *R-1* of α_i and *R* of μ_i

The M/H$_r$/1 queue

- If there are different service needs randomy intermixed
  - E.g., packet size distribution

  or if the service time distribution has $C_x^2>1$ – approximate with H_r
- The state represents the number of customers in the system and the actual server used (only one server used at a time!)
  - complicated Markov-chain (see notes from class)
  - you have to be able to handle it for limited buffer systems
  - Little, PASTA holds
  - Average perfromance with M/G/1



The M/H_r/1 queue

- Example problem: Packets of two types arrive to a multiplexer intermixed. The total arrival intensity is λ.

  Packet of type 1 arrives with probability α_1, its transmission time is exponential with parameter μ_1.

  Packet of type 2 arrives with probability α_2, its transmission time is exponential with parameter μ_2.

  There is no buffer.
- Give:
  - Kendall, Markov-chain
  - state probabilities (balance equations)
  - P(packet type 1 under transmission)
  - P(packet blocked)
  - Utilization

Method of stages for the arrival process

- Non-exponential inter-arrival times can be modeled similarly
- E.g., round-robin customer spreading: $E_r/M/1$

Semi-markovian system
Method of stages - Summary

- Ways to handle the non-exponential service / inter-arrival time
  - Method of stages: look at distributions consisting of several exponentially distributed stages in series or in parallel
  - Describe the system in specific points of time (end of service) – M/G/1, embedded Markov-chains

- Erlang-r service / inter-arrival times
  - series of stages in the real system, or
  - has distribution with $C_x^2 < 1$
  - can be modeled with Markov-chain
    state: number of customers time r plus number of stages left from service
- Hyper-exponential service /inter-arrival times
  - parallel stages in the real system
  - has distribution with $C_x^2 > 1$
  - can be modeled with Markov chain
    state: number of customers and server used