# EP2200 Queueing theory and teletraffic systems

## Lecture 9
## M/G/1 systems

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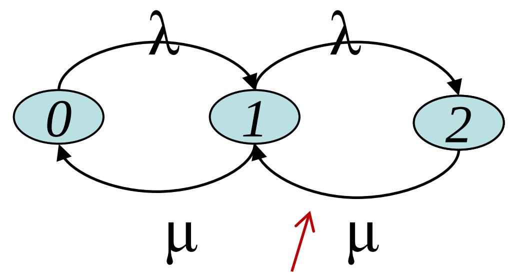
# The M/G/1 queue

- Arrival process memoryless (Poisson($\lambda$))
- Service time general, identical, independent, f(x)
- Single server
- M/$E_r$/1 and M/$H_r$/1 are specific cases, results for M/G/1 can be used

- Rules we can "use" from the Markovian systems
  - $\rho=\lambda E[x] < 1$ for stability (single server, no blocking)
  - Little: $N= \lambda T$
  - PASTA

# The M/G/1 queue

- Recall: M/M/1:




At the arrival of the second customer the
time remaining from the service of the first customer is still Exp(▯)

- M/G/1:
  - If we consider the system when a new customer arrives, then
  - the remaining (residual) service time of the customer under service depends on the past of the process (on the elapsed service time)

- Consequently: the number of customers in the system does not give a Markov chain

# The M/G/1 queue

- Solution methods
  - Average measures N, T, etc.
    - Mean value analysis
  - Distribution of the number of customers, waiting time, etc.
    - Study the system at time points $t_0$, $t_1$, $t_2$, ... when a customer departs, and extend for all points of time
    - Embedded Markov chain

# The M/G/1 mean value analysis
# Pollaczek-Khinchin mean formulas

- To calculate average measures
- We start with the average waiting time:
  - the service of the waiting customers + the remaining (or residual service) of the customer in the service unit
  - Average remaining service time: $R_s$
  - First conditional average waiting time, then uncondition

$$W_k = R_{s,k} + \sum_{i=1}^{k-1} X_i, \quad k \geq 1$$

$$E[W_k] = E[R_{s,k}] + (k-1)E[X], \quad k \geq 1 \quad \text{(average waiting time for customer arriving at state k)}$$

$$W = E[W] = \sum_{k=0}^{\infty} p_k E[W_k] = p_0 0 + \sum_{k=1}^{\infty} p_k E[R_{s,k}] + \sum_{k=1}^{\infty} p_k (k-1) E[X]$$

$$R_s \stackrel{\Delta}{=} \sum_{k=0}^{\infty} p_k E[R_{s,k}], \quad r_{s,0} = 0 \quad \text{(average includes 0 remaining service times at state 0)}$$

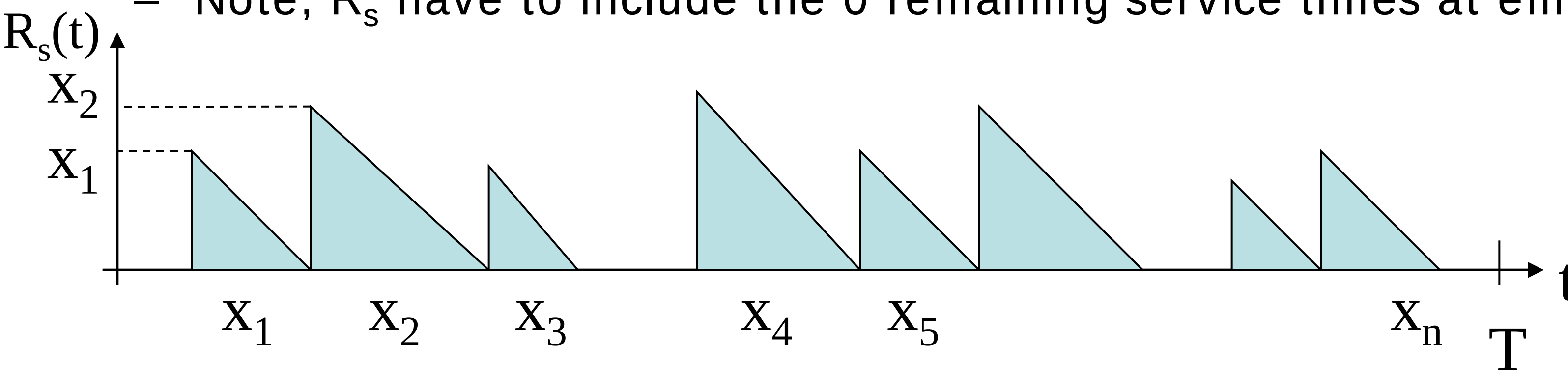
$$W = R_s + N_q E[X]$$

$$W = R_s + W\lambda E[X]$$

$$W = \frac{R_s}{1-\rho}$$

# The M/G/1 mean value analysis Pollaczek-Khinchin mean formulas

- We have to derive the average remaining service time $R_s$ :
  - n: number of services in a large T = number of Poisson arrivals: $n=\lambda T$ (since the system is stable)
  - $T \to \infty$ and $n \to \infty$
  - Note, $R_s$ have to include the 0 remaining service times at empty system.




$$R_s = E\left[R_s(t)\right] = \frac{1}{T}\sum_{i=1}^{n}\frac{1}{2}X_i^2 = \frac{\lambda}{n}\frac{1}{2}\sum_{i=1}^{n}X_i^2 = \frac{\lambda}{2}E[X^2]$$

$$W = \frac{R_s}{1-\rho} = \frac{\lambda E[X^2]}{2(1-\rho)}$$

Pollaczek-Khinchin mean formula for the waiting time

# The M/G/1 mean value analysis
# Pollaczek-Khinchin mean formulas

- From W you can derive T, N, $N_q$ with Little's theorem
- Comments:
  - W depends on the first and the second moment of the service time only
  - Mean values increase with variance (cost of randomness)

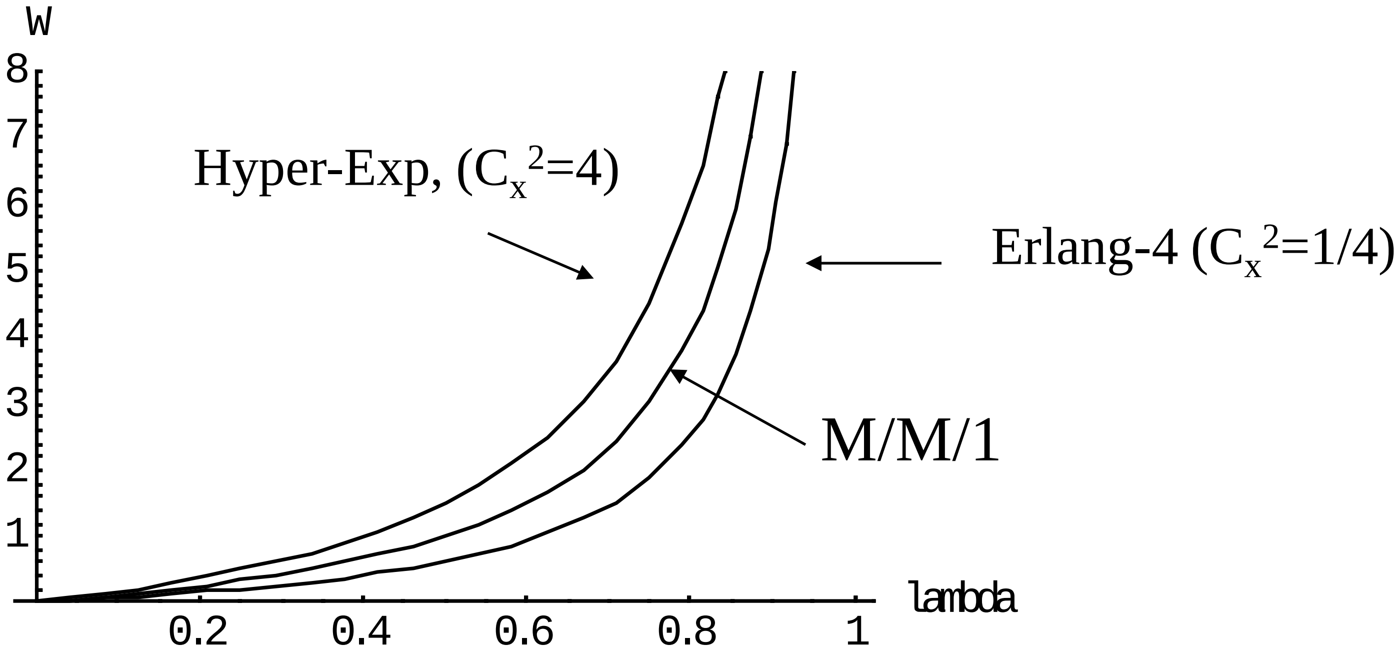
$$W = \frac{\lambda E[X^2]}{2(1-\rho)} = \frac{\lambda(E[X]^2 + V[X])}{2(1-\rho)} = \frac{\lambda(E[X]^2 + V[X]\frac{E[X]^2}{E[X]^2})}{2(1-\rho)} = \frac{\rho E[X]}{2(1-\rho)}(1+C_x^2)$$

$$M/M/1: \quad C_x^2 = 1, \quad W = \frac{\rho E[X]}{(1-\rho)}$$

$$M/D/1: \quad C_x^2 = 0, \quad W = \frac{\rho E[X]}{2(1-\rho)}$$

# M/G/1 waiting time

$$W = \frac{\rho E[X]}{2(1-\rho)}(1 + C_x^2)$$



# The M/G/1 mean value analysis
# Pollaczek-Khinchin mean formulas

Group work:

- Consider the following system:
  - Single server, infinite buffer
  - Poisson arrival process, 0.1 customer per minute
  - Service process: sum of two exponential steps, with mean times 1 minute and 2 minutes.
  - Calculate the mean waiting time

$$W = \frac{\lambda E[x^2]}{2(1-\rho)} = \frac{\rho E[x]}{2(1-\rho)}(1+C_x^2)$$

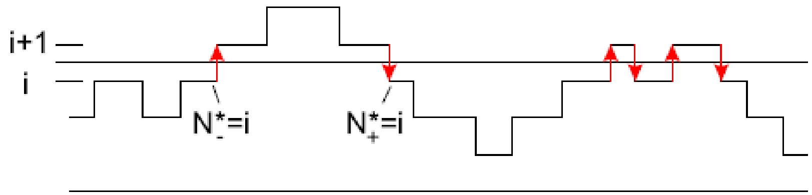
# Distribution of number of customers in the system

*** Comment: called as queue-length in the Virtamo notes! ***

- The number of customers, $N_t$ is not a Markov process
  - the residual service time is not memoryless
- We can model the system at departure time and extend the results to all points of time:
  - in the case of Poisson arrival the distribution of N at departure times is the same as at arbitrary points of time (PASTA)
  - if we are lucky then $N_t$ follows a discrete time Markov process at departure times
  - since this discrete time Markov chain is rather complex, we can express the transform form (z-transform) of the distribution of the number of customers in the system.

# Distribution of number of customers in the system

- In the case of Poisson arrival the distribution of N at departure times is the same as at arbitrary points of time (PASTA)
  - PASTA is proved for arrival instants
  - however, departure instants see the same queue length distribution




- Let us follow $N_k$, $N_{k+1}$, $N_{k+2}$..., that is, the number of customers in the system after departures

  $N_k$: number of customers after the departure of a customer k

  $V_k$: number of arrivals during the service time of customer k,

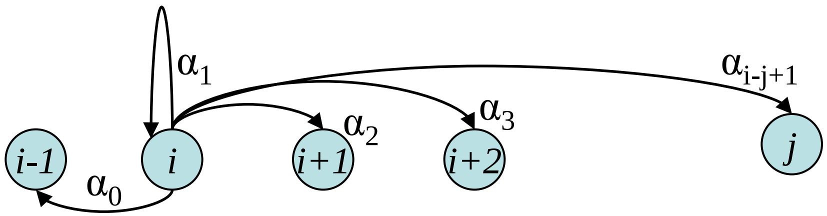
  b(x) is the service time distribution, then:

$$N_{k+1} = \begin{cases} N_k - 1 + V_{k+1} & N_k \geq 1 \\ V_{k+1} & N_k = 0 \end{cases}$$

$\Rightarrow$ $N_{k+1}$ depends only on $N_k$ and V,
V is independent from k
$\rightarrow$ Discrete time Markov Process

$$\alpha_i = P(V = i) = \int \frac{(\lambda x)^i}{i!} e^{-\lambda x} b(x) dx$$

# M/G/1 number of customers in the system




- Expressing the steady state of the Markov-chain describing N, we get the z-transform of the distribution of N
- Pollaczek-Khinchin transform form:

$$Q(z) = B^*(\lambda - \lambda z)\frac{(1-\rho)(1-z)}{B^*(\lambda - \lambda z) - z}$$

  – where: $\rho=\lambda E[X]$ and B*(s) is the Laplace transform of the service time distribution. (S*(s) in the Virtamo notes)

- Distribution of N with inverse transform, or moments of the distribution.
- E.g., M/M/1

# M/G/1 system time distribution

- Without proof:
- Pollaczek-Khinchin transform form for the system time and waiting time:

$$W^*(s) = \frac{s(1-\rho)}{s-\lambda+\lambda B^*(s)}$$

$$T^*(s) = B^*(s)\frac{s(1-\rho)}{s-\lambda+\lambda B^*(s)}$$

  – where: $\rho=\lambda E[x]$ and B*(s) is the Laplace transform of the service time distribution. (S*(s) in the Virtamo notes)
- E.g., M/M/1 system time

# The M/G/1 mean value analysis Pollaczek-Khinchin mean formulas

Group work again:

- Consider the system:
  - Single server, infinite buffer
  - Poisson arrival process, 0.1 customer per minute
  - Service process: sum of two exponential steps, with mean times 1 minute and 2 minutes.
  - Give the Laplace transform of the waiting time, calculate the mean waiting time

$$W^*(s) = \frac{s(1-\rho)}{s-\lambda+\lambda B^*(s)}$$

# M/G/1

- Requirements for the exam:

- Derive and use P-K mean value formulas
- Use P-K transform forms
  - typically: for given service time distribution give the transform forms, calculate moments

- Do not forget: M/M/1, M/D/1, $M/E_r/1$ and $M/H_r/1$ are specific cases of M/G/1