R9 – M/G/1 queues

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In this recitation, we consider M/G/1 queues where the service time could be any kinds of distribution. If we know the average and the variance (or the secondary moment) of the service time, we can get average performance measures of this queuing system such as average waiting time, average number of customers in the system, etc, by using the mean value formulas. However, if we know the Laplace transform of the service time, we can get the Laplace transform of the waiting time and also the Z-transform of the number of customers in the system by using the transform formulas, and then we can get the moments and the distribution functions of these variables based on their transforms. The lecture introduced how we derive these formulas. In this recitation, I will first summarize some preliminary knowledges needed (my note is attached), and then start to solve the following problems to show how can we use these formulas to get different performance measures for M/G/1 queues.

1 Exercise 10.1

Please check the solution in the solution manual. This exercise shows an example on how to use the mean value and also the transform formulas. It also implies that the transforms can give us more information including the average performance, but to derive the transforms is more complicate.

2 Exercise 10.2

Please check the solution in the solution manual. This exercise suggests that the results for M/G/1 queues also hold for M/E_r/1 and M/H_r/1 queues.

3 Exercise 10.4

Please check the solution in the solution manual. The figure at the end of this document will help you to understand the system.

M/G/1 $\quad G \in \{D, M, E_r, H_r, \cdots\}$

① mean value formulas

$$\bar{W} = \frac{\lambda E[S^2]}{2(1-\rho)} = \frac{1+C_S^2}{2} \cdot \frac{\rho}{1-\rho} E[S]$$

$$\bar{N} = \underbrace{\rho}_{\bar{N}_S = \lambda \bar{S}} + \underbrace{\frac{1+C_S^2}{2} \cdot \frac{\rho^2}{1-\rho}}_{\bar{N}_q = \lambda \bar{W}}$$

* $E[S], E[S^2]$

$V[S] = E[S^2] - E[S]^2$

$C_S^2 = \frac{V[S]}{E[S]^2}$

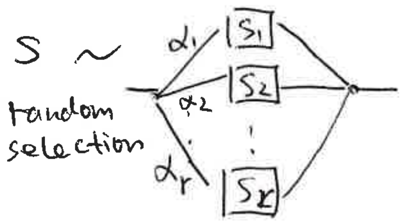
$\sigma_S = \sqrt{V[S]}$

Ex - 1

① $S = S_1 + S_2 + \cdots S_r$ $\quad$ S_i ~ independent variables

We know $E[S_i], E[S_i^2], V[S_i]$

how to express $E[S], E[S^2], V[S]$?

② What about if $S \sim$ 

$Pr(S \sim S_i) = \alpha_i$?

Solution:

① $E[S] = \sum_{i=1}^{r} E[S_i]$, $\quad V[S] = \sum_{i=1}^{r} V[S_i]$ $\Rightarrow E[S^2] = V[S] + E[S]^2$

② $E[S] = \sum_{i=1}^{r} \alpha_i E[S_i]$, $\quad E[S^2] = \sum_{i=1}^{r} \alpha_i E[S_i^2]$ $\Rightarrow V[S] = E[S^2] - E[S]^2$

② Transform formulas:

preliminary knowledge:

△ $x \sim$ discrete random variable

z transform of x: $G_x(z) = \sum_{i=1}^{\infty} P_i z^i = E[z^x]$

$$E[x^i] = \frac{d}{dz}\underbrace{\left(z\frac{d}{dz}\right)^{i-1} G_x(z)}\Bigg|_{z=1}$$

$$\underbrace{\frac{d}{dz}\left(\underbrace{z\frac{d}{dz}\cdots\left(\frac{d}{dz}z\left(\frac{d}{dz}G_x(z)\right)\right)}_{i-1}\right)}_{i}$$

$$* \quad \frac{d\,G_x(z)}{dz}\bigg|_{z=1} = E[x z^{x-1}]_{z=1} = E[x]$$

△ $x \sim$ contineous random variable

Laplace transform

$$f_x^*(s) = \int_0^{\infty} e^{-sx} f(x)\,dx = E[e^{-sx}]$$

$$f_{x+y}^*(s) = E[e^{-s(x+y)}] = E[e^{-sx}]\cdot E[e^{-sy}]$$

$$= f_x^*(s)\cdot f_y^*(s)$$

$$E[x^n] = (-1)^n f^{*(n)}(0)$$

$$E[x] = -f^{*'}(0), \quad E[x^2] = +f^{*''}(0)$$

$$f(x) = \mathcal{L}^{-1}(f_x^*(s))$$

$x \sim Exp(\lambda)$ $\quad f_x^*(s) = \frac{\lambda}{s+\lambda}$

$x \sim \delta(t)$ $\quad f_x^*(s) = 1$

in M/G/1

$$G_N(z) = \frac{(1-\rho)(1-z)}{1 - \dfrac{z}{S^*(\lambda(1-z))}}$$

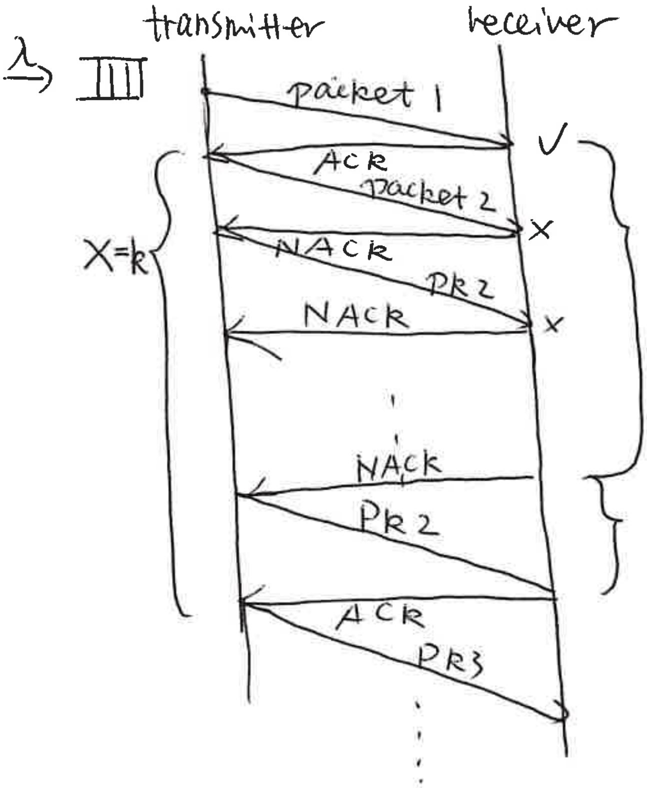
$S^*(\lambda(1-z)) = S^*(s)\big|_{s=\lambda(1-z)}$

$S^*(s)$: Laplace transform of service time

$$W^*(s) = \frac{(1-\rho)s}{s-\lambda+\lambda S^*(s)}$$

$$T^*(s) = W^*(s)\cdot S^*(s) \quad \text{since } T = W + S$$

10.4



k-1 times : not successful

last time : successful

$$\therefore P(X=k) = \underbrace{p^{k-1}}_{\text{prob. for error transmission in } k-1 \text{ times.}} \cdot \underbrace{(1-p)}_{\text{prob that the last time is successful}}$$

$X \sim \text{geometric}(1-p)$

$$\overline{X} = \frac{1}{1-p}, \quad V[X] = \frac{1-p}{p^2}$$