R8 – Stage method, the Erlang and Hyper-exponential distributions

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November 28, 2012

From this chapter, we start to look at semi-Markovian system. In this chapter, we consider two spatial cases: $M/E_r/1$ and $M/H_r/1$, where the service time is not exponentially distributed, but Erlang and Hyper-exponentially distributed, respectively.

From exercise 9.1 and 9.2, you will learn how to use the method of stages to build the Markov chain for $M/E_r/1$ and $M/H_r/1$, respectively, and how to get the state probabilities and other performance measures for limited buffer systems.

Two extra exercise problems are given in the recitation. They will very helpful to improve your understanding. If you miss this recitation, I could suggest you to solve these four problems by yourself, and then discuss your solutions of the two extra exercise problems with your classmates who have attended this recitation or ask me directly.

1 Exercise 9.1

Consider an $M/E_r/1$ system without buffer. Let the state j be the number of stages left of service, and P_j be the equilibrium probability that the system is in state j.
a) Find the value of P_j for $j = 0, 1, ...r$.
b) Find the probability of a busy system.

Solution: Please check the solution in the solution manual.

2 Exercise 9.2

Consider an $M/H_2/1$ system, without buffer. The branching probabilities are α and $1 - \alpha$ to servers 1 and 2, and the service rates are $\mu_1 = 2\mu\alpha$ and $\mu_2 = 2\mu(1 - \alpha)$.
a) Calculate the steady state probability that the system is idle.
b) What is the probability of the first service unit being busy?
c) What is the probability that the system is busy?

Solution: Please check the solution in the solution manual.

3 Extra Exercise 1

Consider four queuing systems with Poisson arrivals and different service time distributions. The average arrival rate is λ. Let x denote the service time. Every systems has only one place in its queue. The number of server and distribution of service time of each system are

System I: 1 server, $x \sim \mathrm{Exp}(\mu)$.

System II:1 server, $x \sim \mathrm{Erlang}_2$, $\bar{x} = \mu$.

System III:1 server, $x \sim$ Hyper-exponential$_2$, $\Pr(x \sim Exp(\mu_1)) = \alpha$ and $\Pr(x \sim Exp(\mu_1)) = 1 - \alpha$.

System IV: 2 servers, $x_1, x_2 \sim Exp(\mu)$.

a) Give the Kendall notation of each system.

b) Define the states and then draw the Markov chain for each system.

c) If we already knew the steady state probabilities by solving the balance equations, how to express the blocking probability, the average customers in the system, and the average system time?

Solution: The solution is attached.

4 Extra Exercise 2

Please choose the right answer for each statement. Motivate your selection, e.g via giving the coefficient of variation in different cases.

Statement I: An $M/H_r/1$ system is equivalent to an (   ) queuing system if the service time of each branch is identical and independent exponentially distributed.

Statement II: An $M/E_r/1$ is equivalent to an (   ) queuing system if $r \to \infty$

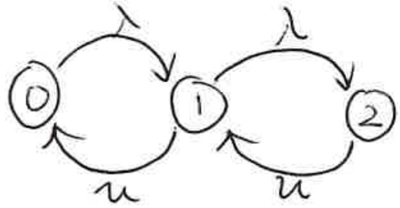
a) $M/M/1$ b) $M/D/1$ c) $M/M/r$

Solution: The solution is attached.

Solution of extra Ex 1:

① $M/M/1/2$

state: number of customers in the system

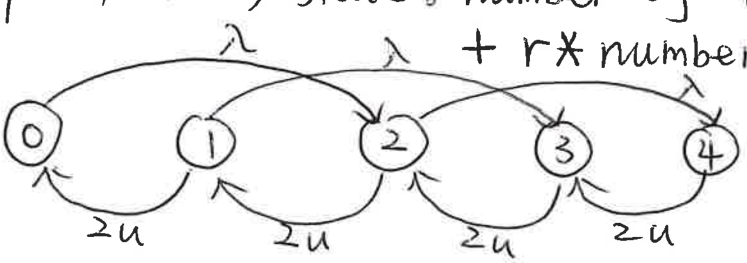


$$P_{blocking} = P_2$$

$$\bar{N} = 0P_0 + 1P_1 + 2P_2$$

$$\bar{T} = \frac{\bar{N}}{\lambda}$$

② $M/E_2/1/2$, state: number of remaining service stages + r * number of waiting customers



$$\frac{1}{u} = \frac{2}{u'}$$

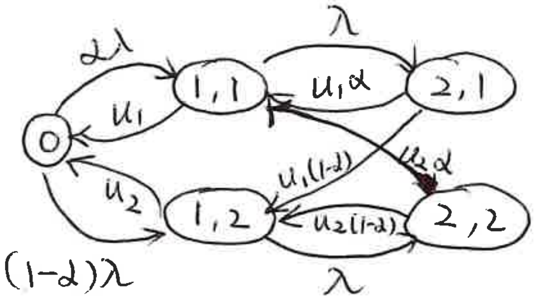
$$u' = 2u$$

$P_{blocking} = P_3 + P_4$, $\bar{N} = 0P_0 + 1(P_1 + P_2) + 2(P_3 + P_4)$

$$N_i = \lceil \frac{i}{r} \rceil$$

$$\bar{T} = \frac{\bar{N}}{\lambda}$$

③ $M/H_2/1/2$, state: {number of customers in the system, current service stage}

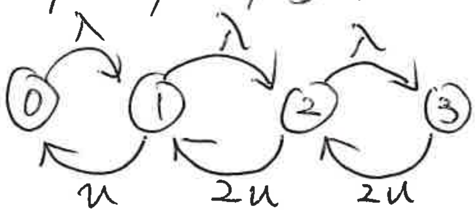


$$P_{blocking} = P_{2,1} + P_{2,2}$$

$$\bar{N} = 0P_0 + 1(P_{1,1} + P_{1,2}) + 2(P_{2,1} + P_{2,2})$$

$$\bar{T} = \frac{\bar{N}}{\lambda}$$

④ $M/M/2/3$, State: number of customers in the system



$$P_{blocking} = P_3$$

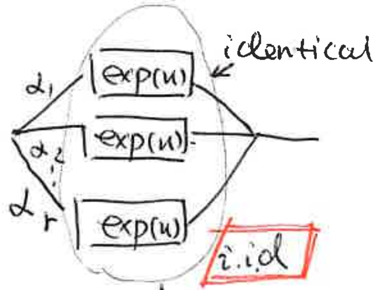
$$\bar{N} = 0P_0 + 1P_1 + 2P_2 + 3P_3$$

$$\bar{T} = \frac{\bar{N}}{\lambda}$$

Solution of extra Ex 2:

- if $x \sim$ Constant, $V[x]=0$, $C_x^2 = \frac{V[x]}{E[x]^2} = 0$
- if $x \sim$ Exp, $V[x] = \frac{1}{u^2}$, $E[x] = \frac{1}{u}$, $C_x^2 = \frac{V[x]}{E[x]^2} = 1$
- if $x \sim$ Erlang-r, $E[x] = \frac{r}{u}$, $V[x] = \frac{r}{u^2}$ u: average service rate of each stage

  $C_x^2 = \frac{1}{r}$
- if $x \sim$ Hyperexponential, and



$$E[x] = \sum \alpha_i E[x_i] = \sum \alpha_i \frac{1}{u_i} = \sum \alpha_i \frac{1}{u} = \frac{1}{u}$$

$$E[x^2] = \sum \alpha_i E[x_i^2] = \sum \alpha_i \frac{2}{u_i^2} = \sum \alpha_i \frac{2}{u^2} = \frac{2}{u^2}$$

$$V[x] = E[x^2] - E[x]^2 = \frac{1}{u^2}, \quad C_x^2 = 1$$

So $M/H_r/1 = M/M/1$ if the service time of each branch is identical and independent exponentially distributed.

$M/Er/1 \sim M/D/1$ if $r \to \infty$.