R11 – Queuing Networks

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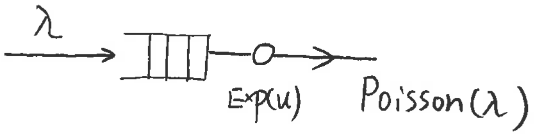
In this recitation, we talk about queuing networks. We show examples to help you understand the concept of open Jackson's queuing network, and also examples (Exercise 12.1, 12.3, and 12.2) on how to analyze the performance measures of a open Jackson's queuing network. The note from this recitation is attached from the next page. If you miss this recitation, please go through the note, discuss the solution of the exercise problems with your classmates or ask me directly if you have any problem.

Queuing Networks

- Properties of Poisson Process

① Departure



$$\mathcal{L}(f_T(t)) = \rho \cdot \frac{u}{s+u} + (1-\rho)\frac{\lambda}{s+\lambda}\cdot\frac{u}{s+u}$$

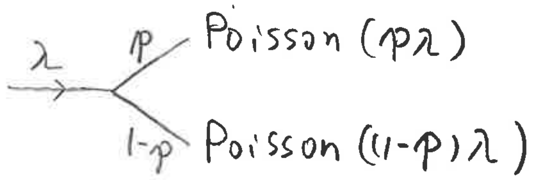
inter-departure time $= \frac{\lambda}{s+\lambda}$

② Superposition

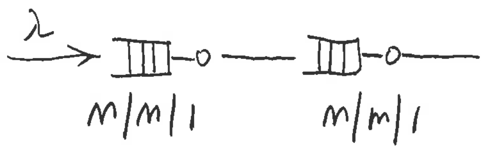
inter arrival time $\sim \min[Exp(\lambda_1), Exp(\lambda_2)]$

$\sim Exp(\lambda_1+\lambda_2)$

③ random split



Tandem System:



$n = \{1, 2, \cdots\}$

- Can be treat as two independent queues
- product form solution

$$P(n_1, n_2) = P(n_1)\, P(n_2)$$

Jackson's queuing network

☆ Conditions need to met please refer to page 6 of slides from the lecture

① $\lambda_j = r_j + \sum_{i=1}^{M} \lambda_i P_{ij}$

② $P(n_1, \dots, n_M) = P_1(n_1)\, P_2(n_2) \cdots P_M(n_M)$

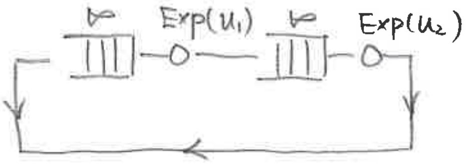
③ Little's theorem applies to the entire network

$N = \sum_{j=1}^{M} N_j$, $T = \frac{N}{\sum_{j=1}^{M} r_j}$ → not λ_j but r_j

④ mean number of nodes a customer visits before leaving $V = \frac{\sum_{j=1}^{M} \lambda_j}{\sum_{j=1}^{M} r_j}$

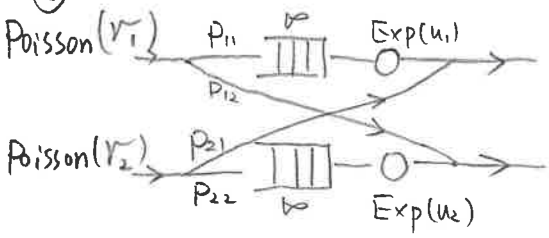
Ex-1, Whether the following queuing networks are Jackson's queuing network? why

①



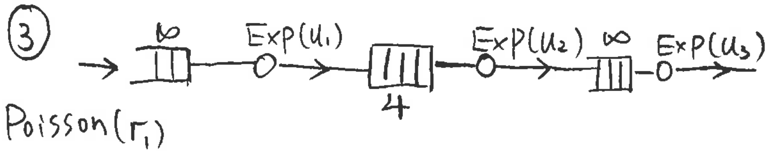
X, not a open queuing network

②



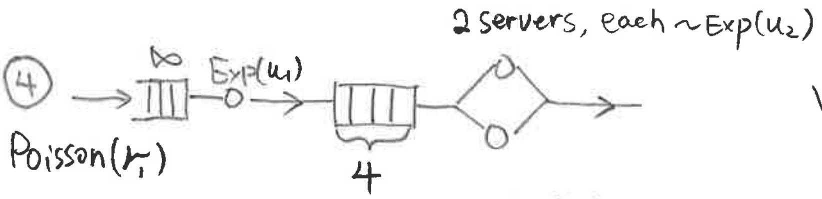
✓, - has poission arrivals
- a departure point is reachable from all arrivals

③



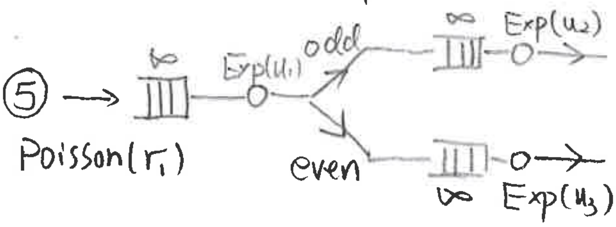
X, the second queue only has finite buffer

④



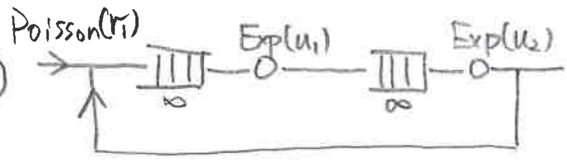
✓, only the 'Last' queue in the network has finite buffer

⑤



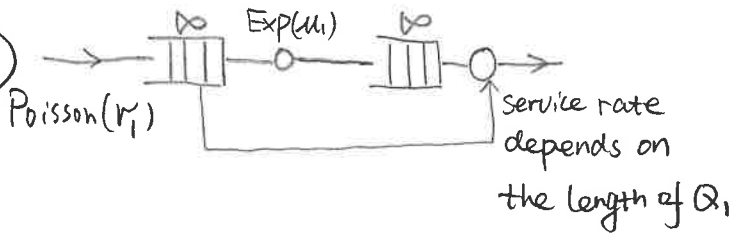
X, not random split

⑥



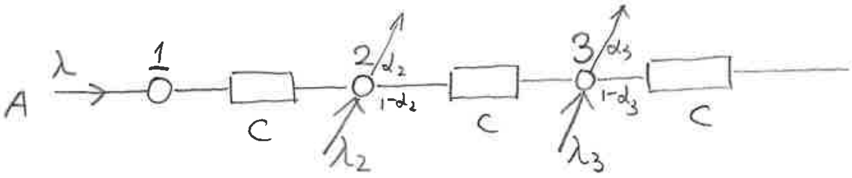
✓, allows feedback

⑦

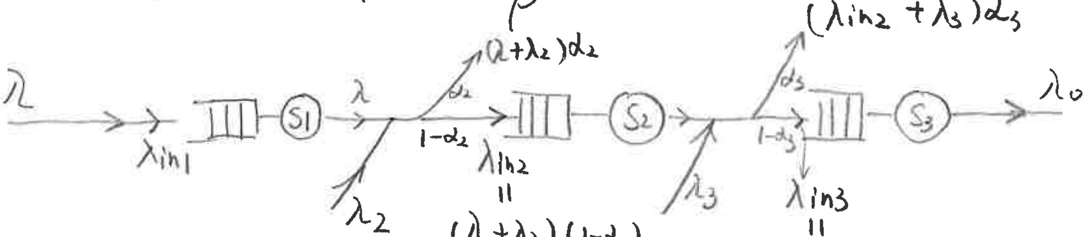


X, service time of Q_2 is not independent

12.1



$\alpha_2 = \alpha_3$, $\quad L \sim \text{Exp}, \bar{L} = \frac{1}{\beta}$



$\lambda_{in2} = (\lambda+\lambda_2)(1-\alpha_2)$

$\lambda_{in1} = \lambda$

$\lambda_{in3} = (\lambda_{in2}+\lambda_3)(1-\alpha_3) = ((\lambda+\lambda_2)(1-\alpha_2)+\lambda_3)(1-\alpha_3)$

$$S_1 = S_2 = S_3 = \frac{L}{C} \sim \text{Exp} \qquad \bar{S} = \frac{\bar{L}}{C} = \frac{1}{\beta C} \triangleq \frac{1}{u_i}$$

a) ρ_1, ρ_2, ρ_3 ☆ Give $\lambda_{in1}, \lambda_{in2}, \lambda_{in3}$ by themselves.

$$\rho_1 = S_1 \lambda_{in1} = \frac{\lambda}{\beta C}$$

$$\rho_2 = S_2 \lambda_{in2} = \frac{1}{\beta C}(\lambda+\lambda_2)(1-\alpha_2)$$

$$\rho_3 = S_3 \lambda_{in3} = \frac{1}{\beta C}(\lambda_3 + (\lambda_1+\lambda_2)(1-\alpha_2))(1-\alpha_3)$$

b) $\bar{N} = \sum_{i=1}^{3} \bar{N}_i$

$\bar{N}_i = \frac{\rho_i}{1-\rho_i}$

$$\left.\right\} \bar{N} = \sum_{i=1}^{3} \frac{\rho_i}{1-\rho_i}$$

c) $\bar{T}_{AB} = \sum_{i=1}^{3} \bar{T}_i$

$$\bar{T}_i = \bar{W}_i + \bar{S}_i = \frac{\rho_i}{(u_i - \lambda_{ini})} + \frac{1}{u_i} = \frac{1}{(\beta C - \lambda_{ini})}$$

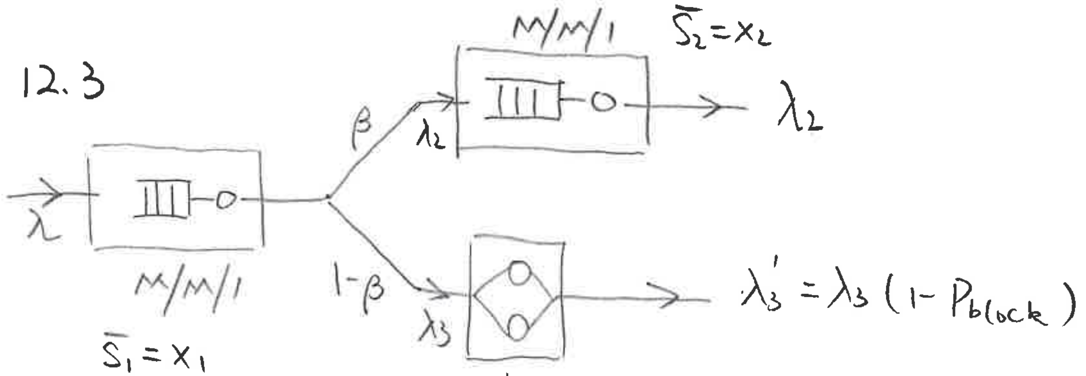
— Can we use

$$\bar{T}_{AB} = \frac{N}{\lambda_1+\lambda_2+\lambda_3} \; ?$$

no $\bar{T} = \frac{N}{\lambda_1+\lambda_2+\lambda_3}$ is the system time of an arbitrary packet including the ones leave at node 2 and 3.

$\bar{T}_{AB} > \bar{T}$

12.3



M/M/2/2 , Erlang loss system

$\lambda = 4 \ \frac{\text{customers}}{\text{min}}$ $\quad X_1 = 10 \text{ sec} \quad X_2 = 30 \text{ sec} \quad X_3 = 20 \text{ sec} \quad \beta = 0.2$

a) λ_2 , λ_3

$\lambda_2 = \beta\lambda = 0.8 \ \frac{\text{customers}}{\text{min}}$

$\lambda_3 = (1-\beta)\lambda = 3.2 \ \frac{\text{customers}}{\text{min}}$

b) $\overline{N_1} , \overline{N_2}$

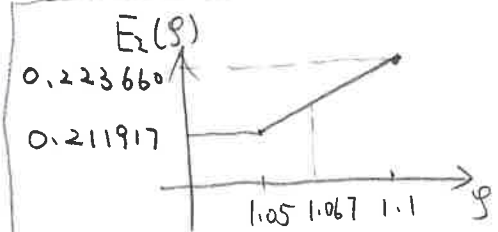
$\overline{N_1} = \frac{\rho_1}{1-\rho_1} = \frac{\lambda x_1}{1-\lambda x_2} = 2$ customers

$\overline{N_2} = \frac{\rho_2}{1-\rho_2} = \frac{\lambda_2 x_2}{1-\lambda_2 x_2} = \frac{2}{3}$ Customers

c) $\lambda_{reject} = \lambda_3 P_{block}$

$P_{block} = E_m(\rho) = E_2(\lambda_3 x_3) = E_3(1.067)$

Erlang-B form



Interpolation

$$\frac{1.067 - 1.05}{1.1 - 1.05} = \frac{x - 0.211917}{0.223660 - 0.211917}$$

$\Rightarrow x \approx 0.2159$

$\lambda_{reject} = 3.2 \times 0.2159$

$\approx 0.67 \ \frac{\text{customers}}{\text{min}}$

d) $\overline{T} = \overline{T_1} + \frac{\lambda_2}{\lambda_2 + \lambda_3'}\overline{T_2} + \frac{\lambda_3'}{\lambda_2 + \lambda_3'}\overline{T_3}$

$= 57.5 s$

$\overline{T_1} = \frac{x_1}{1-\rho_1} = 30 \text{ sec}$

$\overline{T_2} = \frac{x_2}{1-\rho_2} = 50 \text{ sec}$

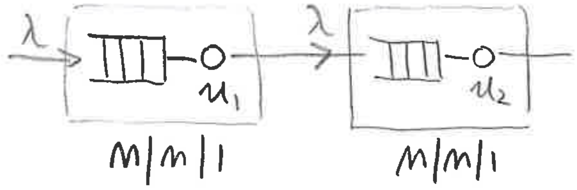
$\overline{T_3} = x_3 = 20 \text{ sec}$

e) $\overline{W} = \overline{W_1} + \frac{\lambda_2}{\lambda_2 + \lambda_3'}\overline{W_2} + 0$

$= 24.8 s$

$\overline{W_1} = \overline{W_2} = \frac{\rho_i}{1-\rho_i}$

12.2



$\mu_1 = \mu_2 = c\beta$

a) $T^*(s)$, $T(t)$

$$T = T_1 + T_2 \Rightarrow T^*(s) = T_1^*(s)\, T_2^*(s)$$

$$T(t) = \mathcal{L}^{-1}(T^*(s))$$

M/m/1: $F_{T_i}(t) = 1 - e^{-(\mu_i-\lambda_i)t} \Rightarrow T_i \sim Exp(\mu_i - \lambda_i)$

$$\Rightarrow T_i^*(s) = \frac{\mu_i - \lambda_i}{s + \mu_i - \lambda_i} \qquad \mu_1 = \mu_2 = c\beta,\ \lambda_1 = \lambda_2 = \lambda$$

$$\Rightarrow T^*(s) = \left(\frac{\beta c - \lambda}{s + \beta c - \lambda}\right)^2 \sim \text{Erlang-2}$$

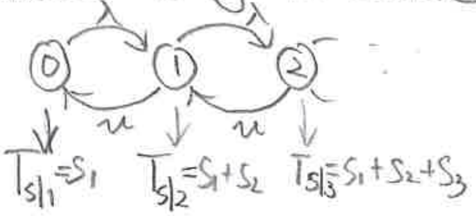
$$T(t) = \begin{cases} (\beta c - \lambda)^2 t e^{-(\beta c - \lambda)t} & t > 0 \\ 0 & t \le 0 \end{cases} \Leftarrow$$

Erlang-r

$$f_x(x, r) = \begin{cases} \lambda^r \frac{x^{r-1}}{(r-1)!} e^{-\lambda x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

$r = 2,\ \lambda = \beta c - \lambda \quad x = t$

☆ another way to derive $T^*(s)$ of M/m/1



$$T_{s|k} = \sum_{i=1}^{k+1} S_i$$

$$T^*(s) = \sum_{k=0}^{\infty} \underbrace{T^*(s \mid \text{already } k \text{ customer in the system})}_{\text{Erlang-k}:\ T^*(s) = \left(\frac{\mu}{s+\mu}\right)^{k+1}} \underbrace{p_k}_{(1-\rho)\rho^k}$$

$$= \sum_{k=0}^{\infty} \left(\frac{\mu}{s+\mu}\right)^{k+1} (1-\rho)\rho^k$$

$$= (1-\rho) \cdot \frac{\mu}{s+\mu} \sum_{k=0}^{\infty} \left(\frac{\lambda}{s+\mu}\right)^k$$

$$= (1-\rho) \frac{\mu}{s+\mu} \frac{1}{1 - \frac{\lambda}{s+\mu}}$$

$$= \frac{\mu - \lambda}{s + \mu - \lambda}$$

b) $E[T]$, $V[T]$

$T \sim \text{Erlang}(2, \beta c - \lambda) \Rightarrow \quad E[T] = \frac{2}{\beta c - \lambda}, \quad V[x] = \frac{2}{(\beta c - \lambda)^2}$

or: $E[T] = -\left.\frac{dT^*(s)}{ds}\right|_{s=0} = \frac{2}{\beta c - \lambda}$

$V[T] = E[T^2] - E[T]^2 = \frac{2}{(\beta c - \lambda)^2}$

$E[T^2] = \left.\frac{d^2 T^*(s)}{ds^2}\right|_{s=0} = \frac{6}{(\beta c - \lambda)^2}$

c) $P(T>t) = 1 - \int_0^t T(t)\,dt \qquad z = \beta c - \lambda$

$= 1 - \int_0^t z^2 x e^{-zx}\,dx$

$u = x, \quad du = dx$
$v = e^{-zx} \quad dv = -z e^{-zx} dx$
$(uv)' = u\,dv + v\,du$

$= 1 + z \int_0^t \underbrace{x}_{u} (\underbrace{-z e^{-zx} dx}_{dv})$

$u\,dv = (uv)' - v\,du$

$= 1 + z \left[uv\Big|_0^t - \int_0^t v\,du \right]$

$= 1 + z\left[te^{-zt} + \frac{e^{zt}}{z} - \frac{1}{z} \right]$

$uv = xe^{-zx}$
$\int_0^t e^{-zx} dx$
$= -\frac{1}{z} e^{-zx}\Big|_0^t$
$= -\frac{1}{z}(e^{-zt} - 1)$

$= zt\,e^{-zt} + e^{zt}$

$= e^{-zt}(zt + 1)$

$= e^{-(\beta c - \lambda)t}\left((\beta c - \lambda)t + 1\right)$

Extra Ex $\quad P_r(2 \text{ packet in the network})$

$= P(n_1 = 2, n_2 = 0) + P(n_1 = 0, n_2 = 2) + P(n_1 = 1, n_2 = 1)$ product form

$= P(n_1=2)P(n_2=0) + P(n_1=0)P(n_2=2) + P(n_1=1)P(n_2=1)$

$= (1-\rho_1)\rho_1^2(1-\rho_2) + (1-\rho_1)(1-\rho_2)\rho_2^2 + (1-\rho_1)\rho_1(1-\rho_2)\rho_2$ $\quad P(n_i = k) = (1-\rho_i)\rho_i^k$

$= 3\left(\frac{\lambda}{c\beta}\right)^2\left(1 - \frac{\lambda}{c\beta}\right)^2$