

EP2200 Queuing theory and teletraffic systems

3rd lecture

Markov chains

Birth-death process

- Poisson process

*Discrete time Markov chains*

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# Outline for today

- Markov processes
  - Continuous-time Markov-chains
  - Graph and matrix representation
- Transient and steady state solutions
- Balance equations – local and global
- Pure Birth process – Poisson process as special case
- Birth-death process as special case
  
- Outlook: Discrete time Markov-chains

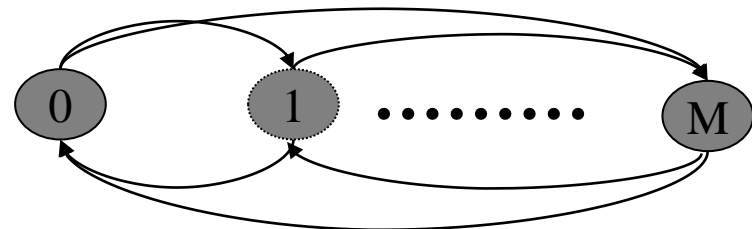
# Markov processes

- Stochastic process
  - $p_i(t) = P(X(t) = i)$
- The process is a Markov process if *the future of the process depends on the current state only* - **Markov property**
  - $P(X(t_{n+1}) = j \mid X(t_n) = i, X(t_{n-1}) = l, \dots, X(t_0) = m) = P(X(t_{n+1}) = j \mid X(t_n) = i)$
  - Homogeneous Markov process: the probability of state change is unchanged by time shift, depends only on the time interval

$$P(X(t_{n+1}) = j \mid X(t_n) = i) = p_{ij}(t_{n+1} - t_n)$$

- Markov chain: if the state space is discrete

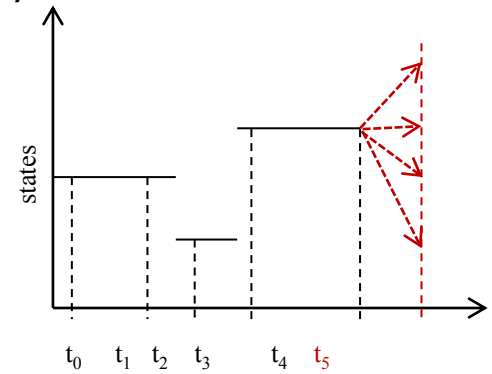
- A homogeneous Markov chain can be represented by a graph:
  - States: nodes
  - State changes: edges



# Continuous-time Markov chains (homogeneous case)

- Continuous time, discrete space stochastic process, with Markov property, that is:

$$P(X(t_{n+1}) = j \mid X(t_n) = i, X(t_{n-1}) = l, \dots, X(t_0) = m) = P(X(t_{n+1}) = j \mid X(t_n) = i), \quad t_0 < t_1 < \dots < t_n < t_{n+1}$$



- State transition can happen in any point of time
- Example:
  - number of packets waiting at the output buffer of a router
  - number of customers waiting in a bank
- **The time spent in a state has to be exponential** to ensure Markov property:
  - the probability of moving from state  $i$  to state  $j$  sometime between  $t_n$  and  $t_{n+1}$  does not depend on the time the process already spent in state  $i$  before  $t_n$ .

# Continuous-time Markov chains (homogeneous case)

- State change probability depends on the time interval:  
 $P(X(t_{n+1})=j \mid X(t_n)=i) = p_{ij}(t_{n+1}-t_n)$
- Characterize the Markov chain with the state **transition rates** instead:

$$q_{ij} = \lim_{\Delta t \rightarrow 0} \frac{P(X(t+\Delta t)=j \mid X(t)=i)}{\Delta t}, \quad i \neq j$$

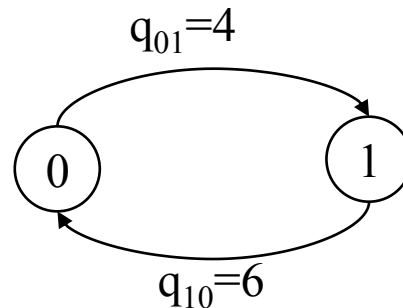
- rate (intensity) of state change

$$q_{ii} = - \sum_{j \neq i} q_{ij}$$

- defined to easy calculation later on

Transition rate matrix  $\mathbf{Q}$ :

$$\mathbf{Q} = \begin{bmatrix} q_{00} & q_{01} & \cdots & q_{0M} \\ \vdots & \ddots & & \\ & & & q_{(M-1)M} \\ q_{M0} & \cdots & q_{M(M-1)} & q_{MM} \end{bmatrix}$$



$$\mathbf{Q} = \begin{bmatrix} -4 & 4 \\ 6 & -6 \end{bmatrix}$$

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- **Transient and steady state solutions**
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# Transient solution

- The transient - time dependent - state probability distribution
- $\underline{p}(t) = \{p_0(t), p_1(t), p_2(t), \dots\}$  - probability of being in state  $i$  at time  $t$ , given  $\underline{p}(0)$ .

$$q_{ij} = \lim_{\Delta t \rightarrow 0} \frac{P(X(t + \Delta t) = j | X(t) = i)}{\Delta t} \Rightarrow P(X(t + \Delta t) = j | X(t) = i) = q_{ij}\Delta t + o(\Delta t)$$

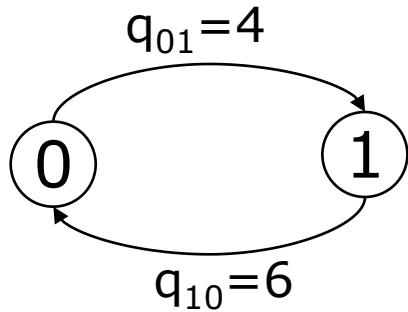
$$p_i(t + \Delta t) = p_i(t) - \underbrace{p_i(t) \sum_{j \neq i} q_{ij} \Delta t}_{\text{leaves the state}} + \underbrace{\sum_{j \neq i} p_j(t) q_{ji} \Delta t}_{\text{arrives to the state}} + o(\Delta t), \quad \lim_{\Delta t \rightarrow 0} \frac{o(\Delta t)}{\Delta t} = 0$$

$$p_i(t + \Delta t) - p_i(t) = p_i(t)q_{ii}\Delta t + \sum_{j \neq i} p_j(t)q_{ji}\Delta t + o(\Delta t) = \sum_j p_j(t)q_{ji}\Delta t + o(\Delta t) \quad \left(-\sum_{j \neq i} q_{ij} = q_{ii}\right)$$

$$\frac{p_i(t + \Delta t) - p_i(t)}{\Delta t} = \sum_j p_j(t)q_{ji} + \frac{o(\Delta t)}{\Delta t} \Rightarrow \frac{dp_i(t)}{dt} = \sum_j p_j(t)q_{ji}$$

$$\frac{d\underline{p}(t)}{dt} = \underline{p}(t)\mathbf{Q}, \quad \underline{p}(t) = \underline{p}(0) \cdot e^{\mathbf{Q}t} \quad \text{Transient solution}$$

# Example – transient solution

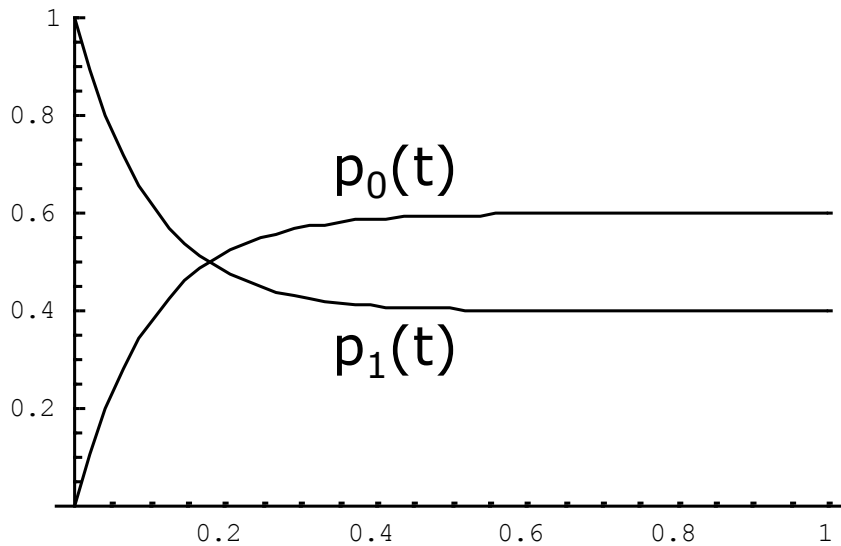


$$Q = \begin{bmatrix} -4 & 4 \\ 6 & -6 \end{bmatrix}$$

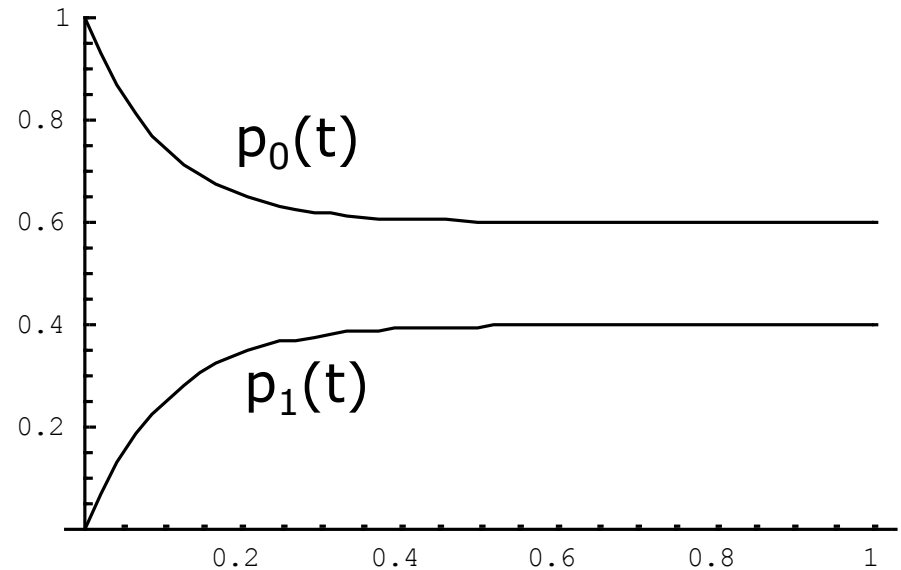
$$p(t)Q = \frac{dp(t)}{dt}$$

$$p(t) = p(0) \cdot e^{Qt}$$

Transient state      Stationary / steady state



A:  $p_0(0)=0, p_1(0)=1$



B:  $p_0(0)=1, p_1(0)=0$



# Transient solution - comment

- Matrix exponential
- Matrix exponentials are defined as:

$$e^{\mathbf{X}} = \sum_{k=0}^{\infty} \frac{1}{k!} \mathbf{X}^k$$

And are difficult to calculate.

- Therefore, for the small case on the 2-state MC I solved the original set of differential equations instead:

$$\frac{dp(t)}{dt} = p(t)\mathbf{Q}$$

$$p_0'(t) = p_0(t)q_{00} + p_1(t)q_{10}$$

$$p_1'(t) = p_0(t)q_{01} + p_1(t)q_{11}$$

$$p_0(t) + p_1(t) = 1$$

- To solve linear differential equations is still tough. See relevant course books.
- Fortunately, math programs can do it for you (I used Mathematica).

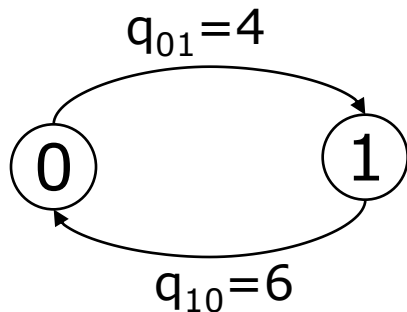
# Stationary solution (steady state)

- Def: stationary state probability distribution (stationary solution)
  - $\underline{p} = \lim_{t \rightarrow \infty} \underline{p}(t)$  exists
  - $\underline{p}$  is independent from  $\underline{p}(0)$
- The stationary solution  $\underline{p}$  has to satisfy:

$$\underline{p}(t)\mathbf{Q} = \frac{d\underline{p}(t)}{dt} = 0, \quad \sum p_i(t) = 1$$

Note: the rank of  $\mathbf{Q}_{MM}$  is  $M-1$ !

$$\mathbf{Q} = \begin{bmatrix} q_{00} & q_{01} & \cdots & q_{0M} \\ \vdots & \ddots & & \\ & & & q_{(M-1)M} \\ q_{M0} & \cdots & q_{M(M-1)} & q_{MM} \end{bmatrix}$$



$$\begin{aligned} [p_0, p_1] \begin{bmatrix} -4 & 4 \\ 6 & -6 \end{bmatrix} &= [0, 0], & p_0 + p_1 &= 1 \\ \hline p_0 &= 0.6, & p_1 &= 0.4 \end{aligned}$$

# Stationary solution (steady state)

Important theorems – without the proof

- Stationary solution exists, if
  - The Markov chain is irreducible (there is a path between any two states) and
  - $\mathbf{p}\mathbf{Q} = 0$ ,  $\mathbf{p} \times \mathbf{1} = 1$  has positive solution
- Equivalently, stationary solution exists, if
  - The Markov chain is irreducible
  - For all states: the mean time to return to the state is finite
- Finite state, irreducible Markov chains always have stationary solution.
- Markov chains with stationary solution are also ergodic:
  - $p_i$  gives the portion of time a single realization spends in state  $i$ , and
  - the probability that one out of many realizations are in state  $i$  at arbitrary point of time

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# Balance equations

- How can we find the stationary solution?  $\underline{p}\mathbf{Q}=\underline{0}$

$$0 = \underline{p}\mathbf{Q} \Rightarrow$$

State 1:

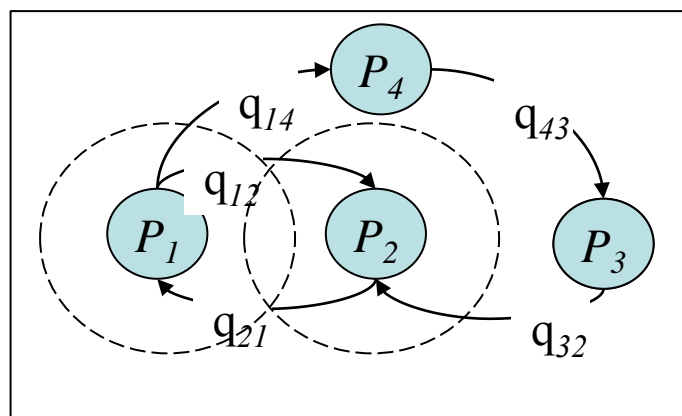
$$0 = -(q_{12} + q_{14})p_1 + q_{21}p_2$$

$$q_{21}p_2 = (q_{12} + q_{14})p_1$$

State 2:

$$0 = q_{12}p_1 - q_{21}p_2 + q_{32}p_3$$

$$\underbrace{q_{12}p_1 + q_{32}p_3}_{\text{flow in}} = \underbrace{q_{21}p_2}_{\text{flow out}}$$



$$\mathbf{Q} = \begin{bmatrix} -(q_{12} + q_{14}) & q_{12} & 0 & q_{14} \\ q_{21} & -q_{21} & 0 & 0 \\ 0 & q_{32} & -q_{32} & 0 \\ 0 & 0 & q_{43} & -q_{43} \end{bmatrix}$$

- Global balance conditions
  - In equilibrium (for the stationary solution)
  - the transition rate out of a state – or a group of states - must equal the transition rate into the state (or states)
    - flow in = flow out
  - defines a global balance equation

# Group work

- Global balance equation for state 1 and 2:

$$0 = p\mathbf{Q} \Rightarrow$$

State 1:

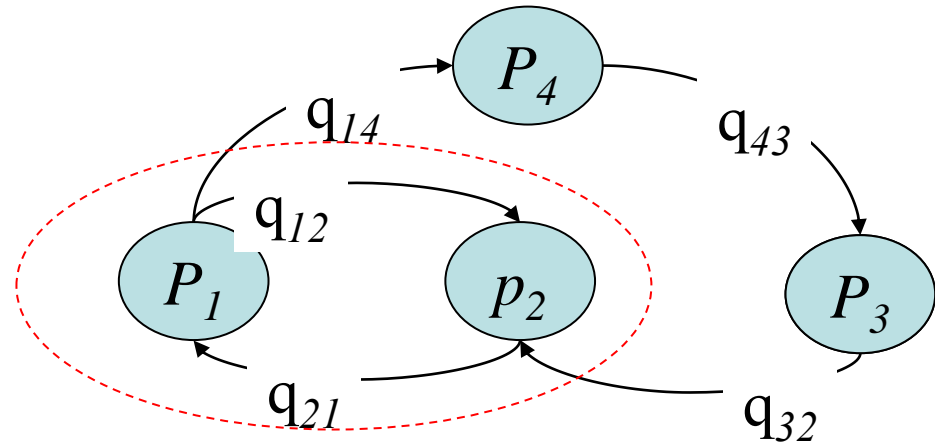
$$0 = -(q_{12} + q_{14})p_1 + q_{21}p_2$$

$$q_{21}p_2 = (q_{12} + q_{14})p_1$$

State 2:

$$0 = q_{12}p_1 - q_{21}p_2 + q_{32}p_3$$

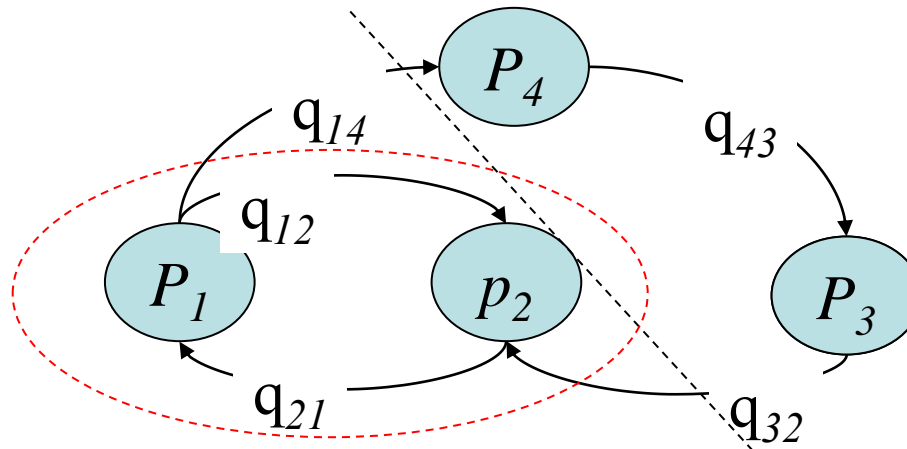
$$q_{12}p_1 + q_{32}p_3 = q_{21}p_2$$



- Is there a global balance equation for the circle around states 1 and 2?

# Balance equations

- Local balance conditions in equilibrium
  - the local balance means that the total flow from one part of the chain must be equal to the flow back from the other part
  - for all possible cuts
  - defines a local balance equation
- The local balance equation is the same as a global balance equation around a set of states!



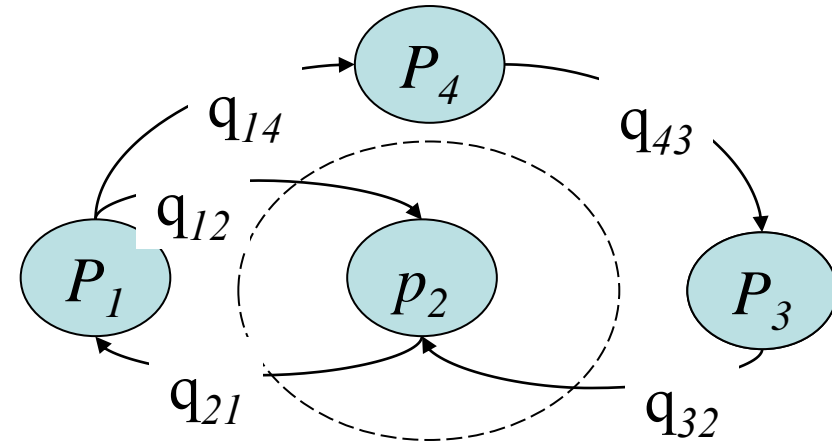
# Balance equations

- Set of linear equations instead of a matrix equation

$$\mathbf{0} = pQ \Rightarrow$$

$$0 = q_{12}p_1 - q_{21}p_2 + q_{32}p_3$$

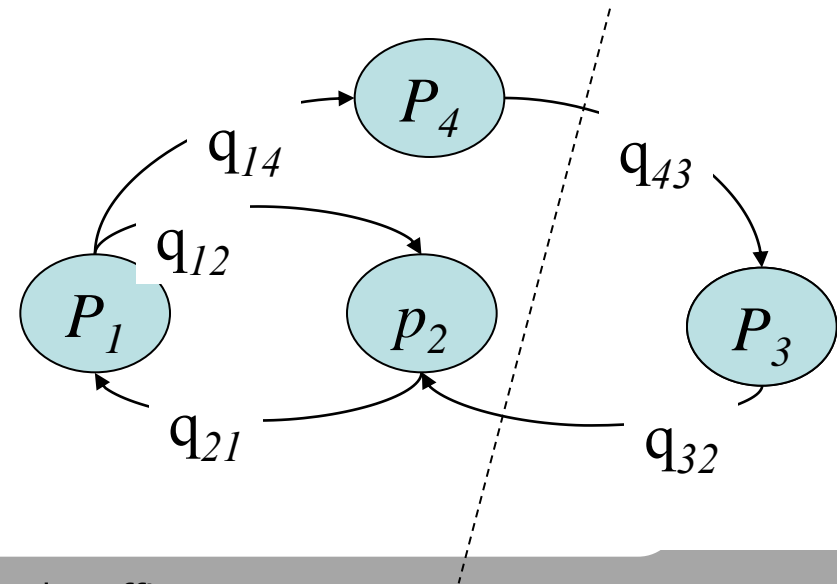
$$\underbrace{q_{12}p_1 + q_{32}p_3}_{\text{flow in}} = \underbrace{q_{21}p_2}_{\text{flow out}}$$



- Global balance :
  - flow in = flow out around a state
  - or around many states
- Local balance equation:
  - flow in = flow out across a cut

$$q_{43}p_4 = q_{32}p_3$$

- M states
  - M-1 independent equations
  - $\sum p_i = 1$



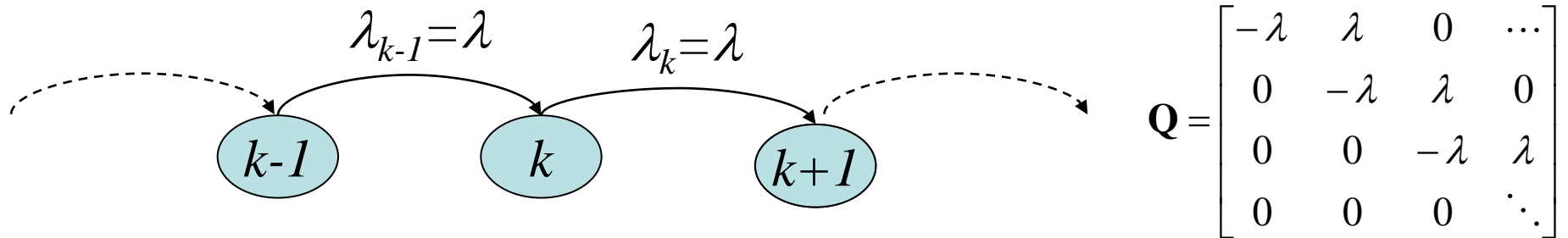


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# Pure birth process

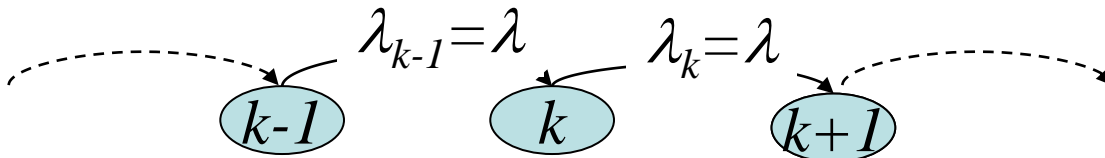
- Continuous time Markov-chain, infinite state space
- Transitions occur only between neighboring states
  - State independent birth intensity:  $\lambda_i = \lambda, \quad \forall i$



- No stationary solution
- Transient solution:
  - $p_k(t) = P(\text{system in state } k \text{ at time } t)$
  - number of events (births) in an interval  $t$

# Pure birth process

- Transient solution – number of events (births) in an interval  $(0,t]$



$$\mathbf{Q} = \begin{bmatrix} -\lambda & \lambda & 0 & \dots \\ 0 & -\lambda & \lambda & 0 \\ 0 & 0 & -\lambda & \lambda \\ 0 & 0 & 0 & \ddots \end{bmatrix}$$

$$\underline{p}'(t) = \underline{p}(t)\mathbf{Q}, \quad p_0(0) = 1, \quad p_k(0) = 0 \quad \text{for } k \neq 0$$

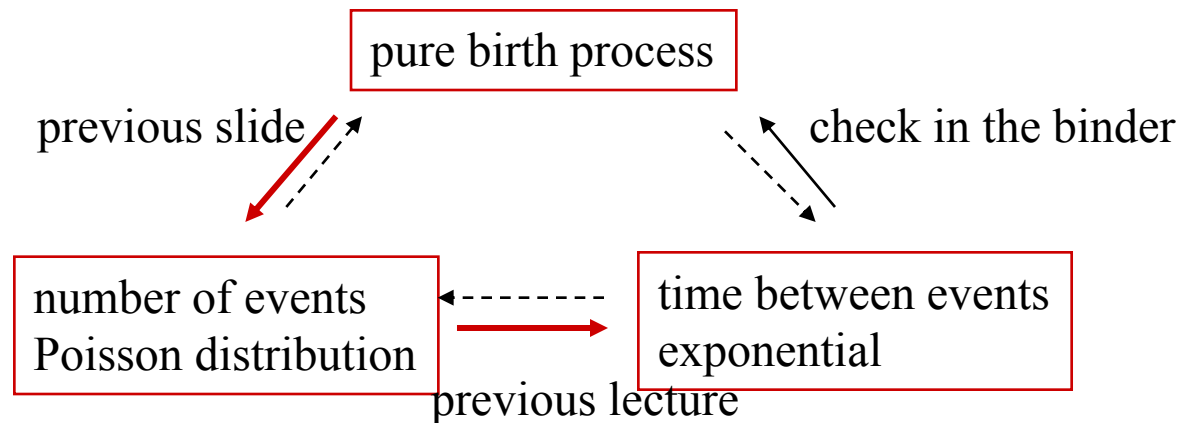
$$\begin{aligned} p_0'(t) &= -\lambda p_0(t) && \rightarrow p_0(t) = e^{-\lambda t} \\ p_1'(t) &= \lambda p_0(t) - \lambda p_1(t) && \rightarrow p_1'(t) = \lambda e^{-\lambda t} - \lambda p_1(t) \rightarrow p_1(t) = \lambda t e^{-\lambda t} \\ &\vdots && \\ p_k'(t) &= \lambda p_{k-1}(t) - \lambda p_k(t) && \Rightarrow p_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \end{aligned}$$

- Pure birth process gives Poisson process! – time between state transitions is  $\text{Exp}(\lambda)$

# Equivalent definitions of Poisson process

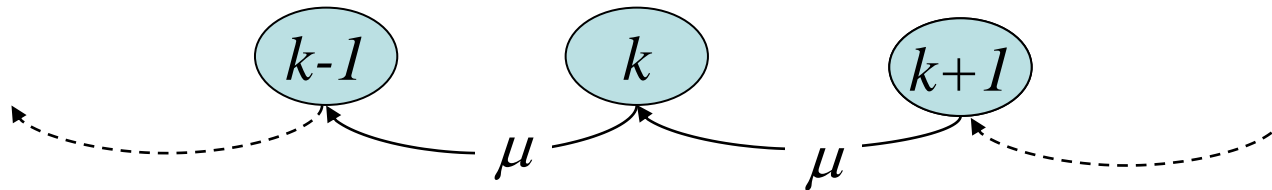
1. Pure birth process with intensity  $\lambda$
2. The number of events in period  $(0,t]$  has Poisson distribution with parameter  $\lambda$
3. The time between events is exponentially distributed with parameter  $\lambda$

$$P(X < t) = 1 - e^{-\lambda t}$$



# Pure death process

- Continuous time Markov-chain, infinite state space
- Transitions occur only between neighboring states
  - State independent death intensity:  $\mu_i = \mu, \quad \forall i \neq 0$



- No stationary solution
- Pure death process gives Poisson process until reaching state 0
- Time between state transitions is  $\text{Exp}(\mu)$

# Outline for today

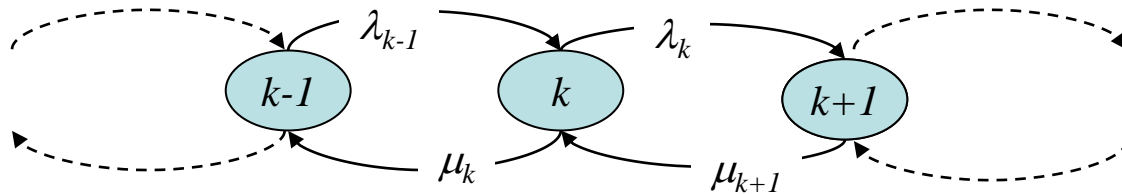
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# Birth-death process

- Continuous time Markov-chain
- Transitions occur only between neighboring states

$i \rightarrow i+1$  birth with intensity  $\lambda_i$   
 $i \rightarrow i-1$  death with intensity  $\mu_i$  (for  $i > 0$ )

} models population



$$\mathbf{Q} = \begin{bmatrix} -\sum q_{0j} & q_{01} & q_{02} & \dots \\ q_{10} & -\sum q_{1j} & q_{12} & \dots \\ q_{20} & q_{21} & -\sum q_{2j} & \dots \\ \vdots & & & \ddots \end{bmatrix} = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & \dots \\ \mu_1 & -(\lambda_1 + \mu_1) & \lambda_1 & 0 \\ 0 & \mu_2 & -(\lambda_2 + \mu_2) & \lambda_2 \\ \vdots & & & \ddots \end{bmatrix}$$

- State holding time – length of time spent in a state  $k$ 
  - Until transition to states  $k-1$  or  $k+1$
  - Minimum of the times to the first birth or first deaths  $\rightarrow$  minimum of two Exponentially distributed random variables:  $\text{Exp}(\lambda_k + \mu_k)$

# B-D process - stationary solution

- Local balance equations, like for general Markov-chains
- Stability: positive solution for  $\underline{p}$  (since the MC is irreducible)

$$\text{Cut 1: } \lambda_{k-1} p_{k-1} = \mu_k p_k \quad \Rightarrow \quad p_k = \frac{\lambda_{k-1}}{\mu_k} p_{k-1}$$

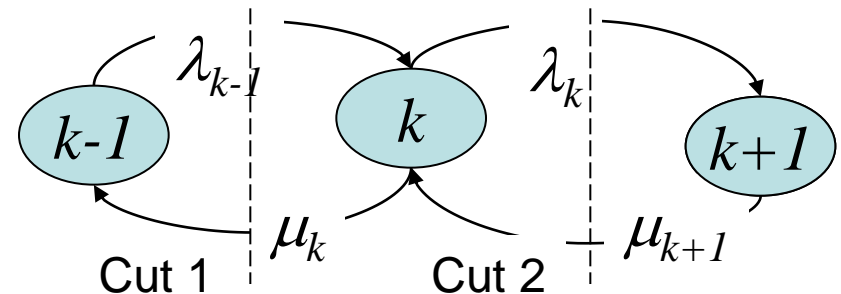
$$\text{Cut 2: } \lambda_k p_k = \mu_{k+1} p_{k+1} \quad \Rightarrow \quad p_{k+1} = \frac{\lambda_k}{\mu_{k+1}} p_k = \frac{\lambda_k \lambda_{k-1}}{\mu_{k+1} \mu_k} p_{k-1}$$

⋮

$$\Rightarrow p_k = \frac{\lambda_0 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_k} p_0 = \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}} p_0,$$

$$\sum p_k = 1 \quad \Rightarrow$$

$$p_0 = \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}},$$



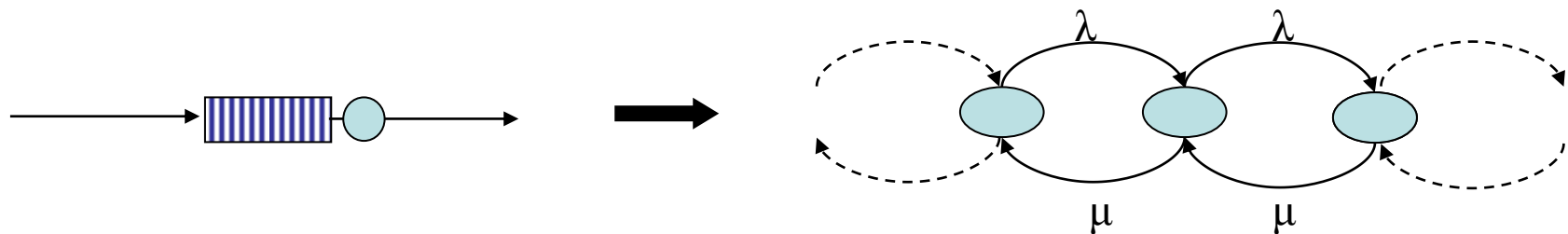
Group work: stationary solution for state independent transition rates:

$$\lambda_i = \lambda, \mu_i = \mu.$$



# Markov-chains and queuing systems

- Why do we like Poisson and B-D processes?  
How are they related to queuing systems?
  - If arrivals in a queuing system can be modeled as Poisson process  $\rightarrow$  also as a pure birth process
  - If services in a queuing systems can be modeled with exponential service times  $\rightarrow$  also as a (pure) death process
  - Then the queuing system can be modeled as a birth-death process

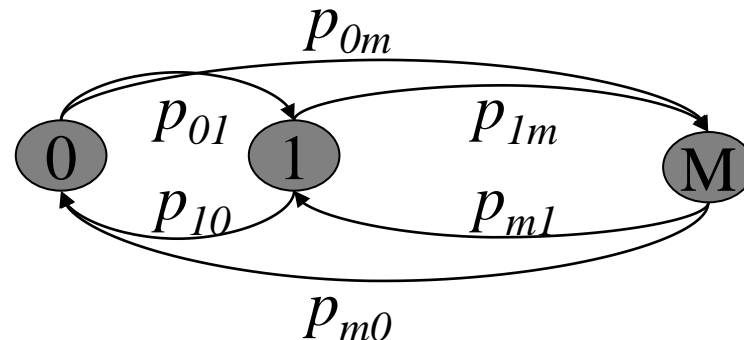


# Summary – Continuous time Markov-chains

- Markovian property: next state depends on the present state only
- State lifetime: exponential
- State transition intensity matrix  $\mathbf{Q}$
- Stationary solution:  $\underline{p}\mathbf{Q}=\underline{0}$ , or balance equations
  
- Poisson process
  - pure birth process ( $\lambda$ )
  - number of events has Poisson distribution,  $E[X]=\lambda t$
  - interarrival times are exponential  $E(\tau)=1/\lambda$
  
- Birth-death process: transition between neighboring states
  
- B-D process may model queuing systems!

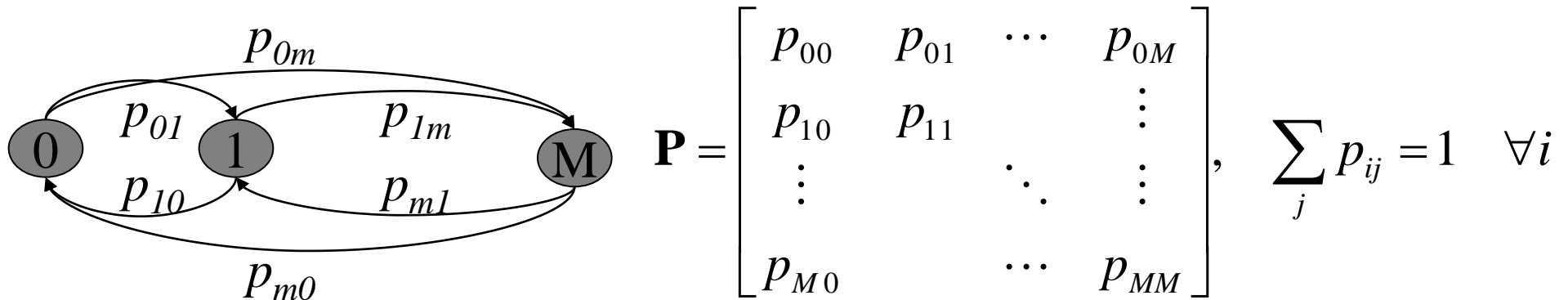
# Discrete-time Markov-chains (detour)

- Discrete-time Markov-chain: the time is discrete as well
  - $X(0), X(1), \dots, X(n), \dots$
  - Single step state transition probability for homogeneous MC:  
 $P(X(n+1)=j \mid X(n)=i) = p_{ij}, \forall n$
- Example
  - Packet size from packet to packet
  - Number of correctly received bits in a packet
  - Queue length at packet departure instants ...  
(get back to it at non-Markovian queues)



# Discrete-Time Markov-chains

- Transition probability matrix:
  - The transitions probabilities can be represented in a matrix
  - Row  $i$  contains the probabilities to go from  $i$  to state  $j=0, 1, \dots, M$ 
    - $P_{ij}$  is the probability of staying in the same state



# Discrete-Time Markov-chains

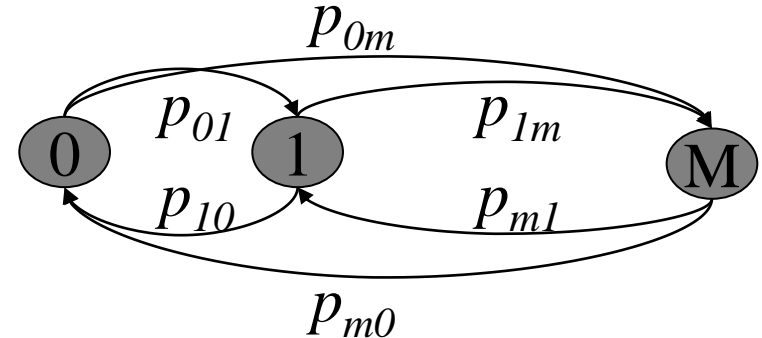
- The probability of finding the process in state  $j$  at time  $n$  is denoted by:
  - $p_j^{(n)} = P(X(n) = j)$
  - for all states and time points, we have:

$$\mathbf{p}^{(n)} = [p_0^{(n)} \quad p_1^{(n)} \quad \cdots \quad p_M^{(n)}]$$

- The time-dependent (transient) solution is given by:

$$p_i^{(n+1)} = p_i p_{ii} + \sum_{j \neq i} p_j^{(n)} p_{ji}$$

$$\mathbf{p}^{(n+1)} = \mathbf{p}^{(n)} \mathbf{P} = \mathbf{p}^{(n-1)} \mathbf{P} \mathbf{P} = \cdots = \mathbf{p}^{(0)} \mathbf{P}^{n+1}$$



# Discrete-Time Markov-chains

- Steady (or stationary) state exists if
  - The limiting probability vector exists
  - And is independent from the initial probability vector

$$\lim_{n \rightarrow \infty} \boldsymbol{p}^{(n)} = \boldsymbol{p} = [p_0 \quad p_1 \quad \cdots \quad p_M]$$

- Stationary state probability distribution is give by:

$$\boldsymbol{p} = \boldsymbol{p} \mathbf{P}, \quad \sum_{j=0}^M p_j = 1 \quad \left( \boldsymbol{p}^{(n+1)} = \boldsymbol{p}^{(n)} \mathbf{P} \right)$$

- Note also:
    - The probability to remain in a state  $j$  for  $m$  time units has geometric distribution
- $$p_{jj}^{m-1} (1 - p_{jj})$$
- The geometric distribution is a memoryless discrete probability distribution (the only one)

# Summary

- Continuous-time Markov chains
- Balance equations (global, local)
- Pure birth process and Poisson process
- Birth-death process
- Discrete time Markov chains