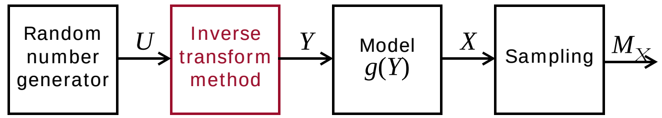
EG2080 Monte Carlo Methods in Engineering

Theme 4

COMPLEMENTARY RANDOM NUMBERS & DAGGER SAMPLING

INTRODUCTION

- All observations in simple sampling are independent of each other.
- Sometimes it is possible to increase the probability of a good selection of samples if the samples are not independent.



MEAN OF TWO ESTIMATES

- Consider two estimates M_{X1} and M_{X2} of the same expectation value μ_X, i.e.,

$$E[M_{X1}] = E[M_{X2}] = \mu_X.$$

- Study the mean of these estimates, i.e.,

$$M_X = \frac{M_{X1} + M_{X2}}{2}.$$

MEAN OF TWO ESTIMATES

- Expectation value:

$$E[M_X] = E\left[\frac{M_{X1} + M_{X2}}{2}\right] =$$

$$= \frac{1}{2}(E[M_{X1}] + E[M_{X2}]) = \frac{1}{2}(\mu_X + \mu_X) = \mu_X, \quad (5)$$

i.e., the combined estimate M_X is also an estimate of μ_X.

MEAN OF TWO ESTIMATES

- Variance:

$$Var[M_X] = Var\left[\frac{M_{X1} + M_{X2}}{2}\right] =$$

$$= \frac{1}{4}Var[M_{X1}] + \frac{1}{4}Var[M_{X2}] +$$

$$+ \frac{1}{4} \cdot 2Cov(M_{X1}, M_{X2}). \quad (6)$$

MEAN OF TWO ESTIMATES

- If M_{X1} and M_{X2} both are independent estimates obtained with simple sampling and the number of samples is the same in both simulations, then we get $Var[M_{X1}] = Var[M_{X2}]$ and $Cov[M_{X1}, M_{X2}] = 0$. Hence, (6) yields

$$Var[M_X] = Var\left[\frac{M_{X1} + M_{X2}}{2}\right] =$$

$$= \frac{1}{4}(Var[M_{X1}] + Var[M_{X1}]) = \frac{Var[M_{X1}]}{2}. \quad (7)$$

MEAN OF TWO ESTIMATES

- The variance obtained in (7) is equivalent to one run of simple sampling using $2n$ samples; according to theorem 11 we should get half the variance when the number of samples is doubled.
- However, if the M_{X1} and M_{X2} are not independent but negatively correlated then the covariance term in (6) will make $Var[M_X]$ smaller than the corresponding variance for simple sampling using the same number of samples.

COMPLEMENTARY RANDOM NUMBERS

- A negative correlation between estimates can be created using complementary random numbers.
- The complementary random number of a $U(0, 1)$-distributed random variable is $U^* = 1 - U$.
  - Correlation coefficient: $\rho_{U, U^*} = -1$.

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COMPLEMENTARY RANDOM NUMBERS

- If a random value is calculated using the inverse transform method, i.e., $Y = F_Y^{-1}(U)$, then the complementary random number is given by $Y^* = F_Y^{-1}(1 - U)$.
  - The complementary random number is calculated in the same way also for normally distributed random numbers generated according to the approximative inverse transform method.
  - For symmetrical distributions we get that if $Y = \mu_Y + \delta$ then $Y^* = \mu_Y - \delta$.
  - Correlation coefficient: $\rho_{Y,\,Y^*} \geq \rho_{U,\,U^*}$.

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COMPLEMENTARY RANDOM NUMBERS

- If a random value is calculated by analysing a scenario, i.e., $X = g(Y)$, then the complementary random number is given by $X^* = g(Y^*)$.
  - Correlation coefficient: $\rho_{X,\,X^*} \geq \rho_{Y,\,Y^*}$.

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ESTIMATES USING COMPLEMENTARY RANDOM NUMBERS

- Let M_{X1} be an estimate based on $x_1, \ldots, x_n$.
- Let M_{X2} be an estimate based on the corresponding complementary random numbers, $x_1^*, \ldots, x_n^*$.
- Obviously M_{X1} and M_{X2} are negatively correlated.

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ESTIMATES USING COMPLEMENTARY RANDOM NUMBERS

- In practice, there is no need to differentiate between the origin of the samples; the final estimate is simply

$$m_X = \frac{1}{2n}\sum_{i=1}^{n}(x_i + x_i^*).$$

MULTIPLE INPUTS

- Generally, the input vector Y has more than one element.
- The complementary random numbers of a vector can be created by combining the original and complementary random numbers of each element in the vector.

  It can be noted that each scenario y_i will have more than one complementary scenario.
- It is possible to apply complementary random numbers only to a selection of the elements in Y.

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EXAMPLE 16 - Complementary random vector

The input vector Y has the three elements A, B, C. The outcomes a_i, b_i, c_i have been randomised for each element, and the corresponding complementary random numbers, a_i*, b_i*, c_i*, have been calculated.

Enumerate the original and complementary scenarios if complementary random numbers are applied to the following elements:

a) A, B, C.

b) A, B.

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EXAMPLE 16 - Complementary random vector

Solution: a) Randomise three values, a, b, c and calculate their complementary random numbers. The results are combined as follows:

$$[a, b, c], \quad [a, b, c*], \quad [a, b*, c], \quad [a, b*, c*],$$
$$[a*, b, c], \quad [a*, b, c*], \quad [a*, b*, c], \quad [a*, b*, c*].$$

b) Randomise six values, a, b, c_1, c_2, c_3, c_4 and calculate the random numbers $a*$ and $b*$. The results are combined as follows:

$$[a, b, c_1], \quad [a, b*, c_2], \quad [a*, b, c_3], \quad [a*, b*, c_4].$$

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EFFECTIVENESS

- The variance reduction effect is depending on the correlation between $x_i = g(y_i)$ and $x_i* = g(y_i*)$. This correlation depends on the correlation between Y and X.
- All inputs may not have a significant correlation to the outputs $\Rightarrow$ no use in applying complementary random numbers to these inputs.
- Sometimes a stronger correlation can be obtained by modifying the input probability distribution.

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AUXILIARY INPUTS

- It might be possible to find a subset of the inputs, Y_i, $i = 1, \ldots, J$ such that there is a strong correlation between $Y_E = h(Y_1, \ldots, Y_J)$ (where h is an arbitrary function) and X.
- When it can be worthwhile to introduce Y_E as an auxiliary input variable to the system, and apply complementary random numbers to Y_E instead of Y_i, $i = 1, \ldots, J$.

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EXAMPLE 17 - Discrete auxiliary input variable

- In example 15, the net income is correlated to the demand.
- The correlation between the demand for an individual flavour and the net income is not as strong as the correlation between the total demand and the net income.
- Hence, we can introduce the auxiliary input

$$D_{tot} = \sum_{f \in \mathcal{F}} D_f.$$

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EXAMPLE 17 - Discrete auxiliary input variables

- We now have to find a joint probability distribution for D_{tot} and D_f, $f = 1, \ldots, 6$.
- The demand for each flavour has three possible states. The total number of combinations is then $3^6 = 729$.
- Enumerate all possible states, calculate the probability for each state and sort the states according to increasing values of D_{tot}.

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EXAMPLE 17 - Discrete auxiliary input variables

- The demand for each flavour can now be randomised with one value from the pseudo-random number generator, U.
- The complementary demand for each flavour is calculated using $U^* = 1 - U$.
- The total demand based on U will be negatively correlated to the total demand based on U^*.

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EXAMPLE 18 - Normally distributed auxiliary input

- Gismo Inc. is selling their products to K retailers, and the demand of each retailer, D_k, is $N(\mu_k, \sigma_k)$-distributed.
- The costs of Gismo Inc. has a stronger correlation to the total demand compared to the demand of an individual retailer.
- Introduce the auxiliary input variable $D_{tot} = D_1 + \ldots + D_K$.

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EXAMPLE 18 - Normally distributed auxiliary input

- The probability distribution of D_{tot} is calculated using the addition theorems for normally distributed random variables.
- To generate the original scenario, randomise D_{tot} as well as temporary values of $D_1, \ldots, D_K$. Then scale the individual retailer demands to match D_{tot}, i.e., let

$$D_k = \frac{D_{tot}}{\sum_j^K D_j'} D_k',$$

where D_k' is the temporary value.

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EXAMPLE 18 - Normally distributed auxiliary input

- To generate the complementary scenario, calculate the complementary random value of D_{tot} and randomise new temporary values of $D_1, \ldots, D_K$. Then scale the individual retailer demands to match D_{tot}^*.

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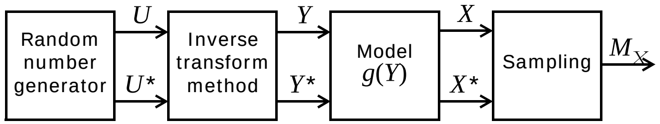
SIMULATION PROCEDURE

- Complementary random numbers



- Step 1. Generate a random number $u_{k,i}$ and calculate $y_i = F_{Yk}^{-1}(u_{i,k})$. If applicable to this scenario parameter, also calculate $y_i^* = F_{Yk}^{-1}(1 - u_{i,k})$.
- Step 2. Repeat step 1 for all scenario parameters. Create all complementary scenarios and calculate $x_i = g(y_i)$.

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SIMULATION PROCEDURE
- Complementary random numbers

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- Step 3. Repeat step 1 and 2 until enough scenarios have been analysed for this batch.
- Step 4. Update the sums

$$\sum_{i=1}^{n} x_i \text{ and } \sum_{i=1}^{n} x_i^2$$

or store all samples x_i (depending on the objective of the simulation).

SIMULATION PROCEDURE
- Complementary random numbers

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- Step 5. Test stopping rule. If not fulfilled, repeat step 1–4 for the next batch.
- Step 6. Calculate estimates and present results.

EXAMPLE 19 - Effectiveness of ICC simulation

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Consider the same system as in example 15. Compare the true expectation values to the probability distribution of estimates from a complementary random number simulation using 200 samples.

EXAMPLE 19 - Effectiveness of ICC simulation

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Table 6 Results of ICC simulation in example 19.

Simulation method	Expected net income [€/day]			Risk for missed delivery [%]			Average simulation time [h:min:s]
	Min	Av.	Max	Min	Av.	Max	
Enumeration		874			0.32		0:34:54
Simple sampling	829	872	909	0.00	0.27	1.00	0:02:25
Complementary random numbers	842	875	914	0.00	0.27	1.00	0:02:25

DAGGER SAMPLING

- We have seen that a negative correlation between the outputs from different scenarios can have a variance reducing effect.
- When the input is a two-state probability distribution, we can create a negative correlation by using dagger sampling instead of complementary random numbers.

DAGGER SAMPLING

- Consider a two-state input variable Y with the frequency function

$$f_Y(x) = \begin{cases} 1-p & x = \text{A}, \\ p & x = \text{B}, \\ 0 & \text{all other } x, \end{cases} \qquad (8)$$

where $p < 0.5$.
- In dagger sampling, the inverse transform method is replaced by a dagger transform function.

DAGGER TRANSFORM

Definition 13: Assume that Y is distributed according to (8) and let S be the largest integer such that $S \leq 1/p$.* Then, Y_i, $i = 1, \ldots, S$, is randomised according to

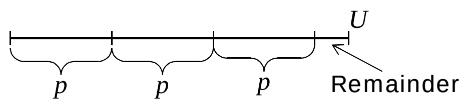
$$F^{\ddagger}_{Yi}(U) = \begin{cases} \text{B} & \text{if } (i-1)p \leq U < ip, \\ \text{A} & \text{otherwise,} \end{cases}$$

where U is a random value from a $U(0, 1)$-distribution.

* The value of S is referred to as the *dagger cycle length*.

DAGGER TRANSFORM

- The dagger transform can be illustrated by dividing an interval between 0 and 1 in S sections, where each section has the width p.
- There will also be a remainder interval if $S \cdot p < 1$.



- Each value of U generates S values of Y.
- y_k = B if U is found in the k:th section, and all other y_i = A, $i = 1, \ldots, S$, $i \neq k$.

EXAMPLE 20 - Dagger transform

The random variable Y is either equal to 2 (70% probability) or 5 (30% probability).

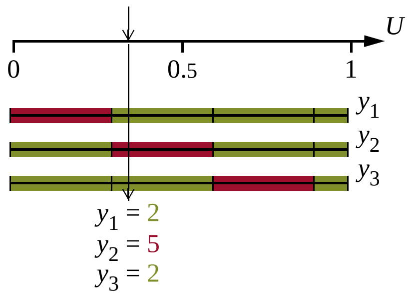
a) Randomise three values of Y using dagger sampling and the random value $U = 0.34$ from a $U(0, 1)$-distribution.

b) Randomise three values of Y using dagger sampling and the random value $U = 0.94$ from a $U(0, 1)$-distribution.

EXAMPLE 20 - Dagger transform

Solution: Here $p = 0.3 \Rightarrow S = 3$. Moreover, we have A = 2 and B = 5.

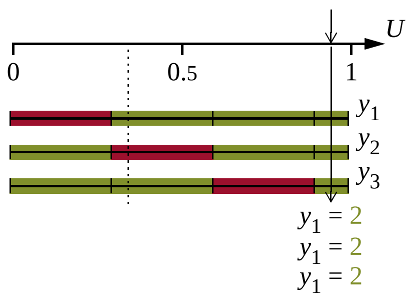
a)



EXAMPLE 20 - Dagger transform

Solution (cont.)

b)



CORRELATION OF DAGGER SAMPLES

- The outputs $X_1, \ldots, X_S$ from the scenarios generated in a dagger cycle will be negatively correlated if the inputs $Y_1, \ldots, Y_S$ are negatively correlated (but the correlation might be weaker).
- Are the input values $Y_i = F^{\ddagger}_{Yi}(U)$, $i = 1, \ldots, S$ negatively correlated, i.e., what is the value of

$$Cov[Y_i, Y_j] = E[Y_iY_j] - E[Y_i]E[Y_j]?$$

CORRELATION OF DAGGER SAMPLES

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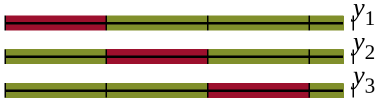
- There are two possible values for the product Y_iY_j: AB or AA. There are two sections there either Y_i or Y_j will be transformed to B; hence, the probability for this event is $2p$. Thus, we get

$$E[Y_iY_j] = 2p\text{AB} + (1 - 2p)\text{AA}.$$



CORRELATION OF DAGGER SAMPLES

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- Moreover, we have

$$E[Y_i] = E[Y_j] = (1 - p)\text{A} + p\text{B}.$$

- The covariance can now be calculated as

$$Cov[Y_i, Y_j] = E[Y_iY_j] - E[Y_i]E[Y_j] =$$
$$= 2p\text{AB} + (1 - 2p)\text{AA} - ((1 - p)\text{A} + p\text{B})^2 =$$
$$= \ldots = = -p^2(\text{A} + \text{B})^2 < 0.$$

- The input values are negatively correlated and we can therefore expect a negative correlation in the outputs too.

MULTIPLE INPUTS

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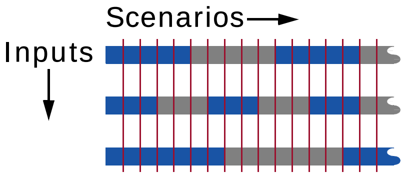
Assume that a computer simulation has several two-state input variables. Which dagger cycle length should be chosen?

- Alternative 1: Independent dagger cycles.



MULTIPLE INPUTS

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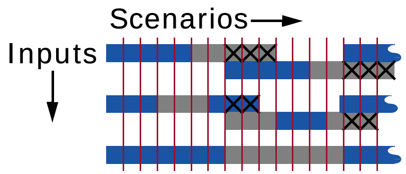
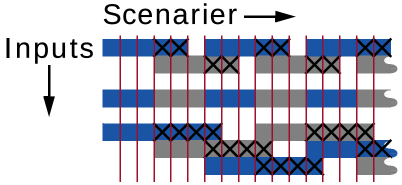
- Alternative 2: Reset all dagger cycles at the end of the longest cycle.

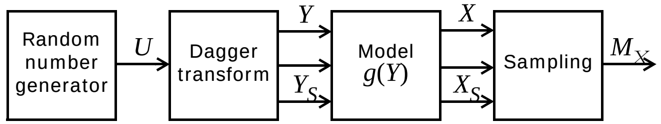


MULTIPLE INPUTS

- Alternative 3: Reset all dagger cycles at the end of the shortest cycle.



SIMULATION PROCEDURE
- Dagger sampling



- Step 1. Generate a random number $u_{k,\,i}$ and calculate $y_i = F^{\ddagger}_{Yi}(u_{i,\,k})$ for one dagger cycle.
- Step 2. Repeat step 1 for all scenario parameters. Combine the dagger cycles and calculate $x_i = g(y_i)$.

SIMULATION PROCEDURE
- Dagger sampling

- Step 3. Repeat step 1 and 2 until enough scenarios have been analysed for this batch.
- Step 4. Update the sums

$$\sum_{i=1}^{n} x_i \text{ and } \sum_{i=1}^{n} x_i^2$$

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SIMULATION PROCEDURE
- Dagger sampling

- Step 5. Test stopping rule. If not fulfilled, repeat step 1–4 for the next batch.
- Step 6. Calculate estimates and present results.

EXAMPLE 21 - Effectiveness of ICC simulation

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Consider the same system as in example 15. Compare the true expectation values to the probability distribution of estimates from a dagger sampling simulation using 200 samples.

EXAMPLE 21 - Effectiveness of ICC simulation

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Table 7 Results of ICC simulation in example 21.

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Compl. r.n.	842	875	914	0.00	0.27	1.00	0:02:25
Dagger sampling	856	877	900	0.00	0.27	1.00	0:02:27

VARIANCE REDUCTION

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- To achieve a variance reduction, we need to have some information of the simulated system.
- For complementary random numbers, we need to identify the inputs where a negative correlation between scenarios will result in negative correlations for one or more outputs.
- For dagger sampling, we need to identify two-state inputs where a negative correlation between the scenarios will result in negative correlations for one or more outputs.