EP2200 Queuing Theory and Teletraffic Systems
Saturday, December 15$^{\text{th}}$, 2012, 09.00-14.00, Q11,Q13, Q15.
Available teachers: Ioannis Glaropoulos, Liping Wang
Allowed help: Calculator, Beta mathematical handbook or similar, provided sheets of queuing theory formulas and Laplace transforms and Erlang tables.

1. Consider a system with two servers, S_1 and S_2. Packets arrive to the system following a Poisson Process with intensity λ packets per second. With probabilities, p_1 and p_2 an arbitrary packet is forwarded to server S_1 or S_2, respectively. Each server has a dedicated buffer that can hold at most 10 packets. Packets arriving to a server with a buffer that is full are dropped. Waiting packets are served at each server on a first-come-first-served basis. The service times are exponential with rates μ_1 and μ_2, respectively, for servers S_1 and S_2. Consider the following values: $\lambda = 10^3$ packets/sec, $\mu_1 = 500\text{sec}^{-1}, \mu_2 = 1000\text{sec}^{-1}$, $p_1 = 0.4, p_2 = 0.6$.

a) Calculate the probability that an arbitrary arriving packet is rejected. Consider an arbitrary rejected packet. What is the probability that the packet has been rejected by S_1? (3p)

b) Consider the event that a packet is rejected. What is the probability that the following arrival is rejected as well? (2p)

Consider, now, that there is a common queue for the two servers and the waiting packets are served, on a first-come-first-served basis, by an available server. The buffer capacity is 10.

c) Re-calculate the probability that once a packet is rejected, the following arrival is rejected as well. (You do not need to know the buffer size to answer the question.) (2p)

d) Assume that, upon arrival, a packet finds 6 other packets waiting in the common queue. Derive the probability that the waiting time of this packet is larger than 10msec. (3p)

2. Five Christmas elves (tomtenissar...) are working in Santa's workshop. They prepare and wrap toys, sharing two wrapping machines. They work like this: each of them keeps preparing toys for an exponentially distributed time with an average of 20 minutes. Then they go to the wrapping machines. If there is a free machine, they wrap the toys, this takes an exponentially long time with an average of 5 minutes, and then go back to prepare toys. If both of the machines are occupied, they immediately go back and prepare some more toys, and so on.

a) Give the Kendall notation of the system, and the state transition diagram. Calculate the probability that both of the machines are idle at an arbitrary point of time. Calculate the utilization of a wrapping machine, that is, the part of the time it is used. (3p)

b) Calculate the probability, that an elf going to the machines finds both of them occupied. (2p)

Assume now that elves arriving when both machines are busy stay there waiting. However, they turn back to make toys, if someone is already waiting for the machines. In addition, elves do the wrapping faster if someone is waiting, and the average time of wrapping in this case decreases to 3 minutes.

c) Give the state transition diagram. Calculate the probability that an elf needs to turn back without wrapping. How does this probability change compared to the first case? Motivate the result. (3p)

d) Calculate the average system time, that is, the time an elf spends at the wrapping machines, including waiting and wrapping. (2p)

3. A source node attempts to transmit data packets to a destination node. The capacity of the link between the source and the destination is 20000 bits/s. Packets arrive at the source as a Poisson process with a mean of 10 packets/s, and are waiting in a FIFO queue with infinite capacity if there is a packet on transmission. 40% of packets are type I packets, whereas the remaining packets are type II. The length of every type I packet is exponentially distributed with a mean of 1000 bits, whereas the length of every type II packet is Erlang-2 distributed with a mean of 2000 bits.

a) What is the probability that the source is busy (i.e. transmitting a packet)? Give the probability density function of the waiting time of an arriving packet that finds the source busy but the queue empty. (2p)

c) Calculate the mean delay (waiting time + transmission time) of an arbitrary packet, and of the two types of packets, respectively. (3p)

Assume now the source starts a sleep period when it becomes idle in order to save energy. Every sleep period lasts for exactly 0.1 s. At the end of a sleep period, the source wakes up if there packets waiting in the queue; otherwise it starts another sleep period. The source can save 10 *Joule* if it sleeps for 1 s.

a) Calculate the mean waiting time of a packet that upon arrival finds the source in sleep mode and the queue empty. (2p)

c) Calculate the mean delay of an arbitrary packet, and also the amount of energy the source can save in one hour. How do these two values change if the source increases the length of a sleeping period? Motivate your answer. (3p)

4. Consider an Accident&Emergency (A&E) department at Danderyds Hospital where there is only one doctor on duty. Patients arrive according to a Poisson process with a mean of 5 patients per hour. 80% of the arriving patients have low priority and their treatment time is exponentially distributed with a mean of 5 minutes. The rest of the patients have high priority and their treatment time is exponentially distributed with a mean of 30 minutes. Assume that patients are treated with preemptive-resume priority.

a) What are the mean waiting times for both classes of patients, and the mean waiting time for an arbitrary patient? (3p)

b) Calculate the mean time a high priority patient needs to spend in the A&E department. What is the probability that a high priority patient needs to wait more than 30 minutes? (2p)

c) Consider a low priority patient that upon arrival finds that there is only one high priority and no low priority patients in the A&E department. What is the probability that the treatment of this patient starts immediately after the treatment of the current patient finishes? What is the probability that the treatment of this patient has to be interrupted? (3p)

d) A high priority patient arrives and finds 2 high priority and 4 low priority patients waiting. Calculate the probability that this patient has to wait for more than 1 hour before being treated. (2p)

5. Answer the following problems!

a) Consider an M/M/10/10 system, with $\lambda = 10\text{sec}^{-1}, \mu = 2\text{sec}^{-1}$. Calculate the average duration of a non-blocking period (in seconds), i.e. the continuous time period when the system does not reject arriving customers. (3p)

b) Consider an M/M/2/3 system. Calculate the average waiting time under *last-come-first-served* policies. (Arrivals do not interrupt the ongoing service process.) $\lambda = 10\text{sec}^{-1}$ per customer, $\mu = 10\text{sec}^{-1}$ (3p)

c) Consider a system with three servers S_1, S_2, S_3. Each server has its own infinitely-sized queue. Messages arrive with a Poisson fashion (λ) and are dispatched to the three servers in a *round-robin* fashion (starting from server S_1). Service times are exponential with rates μ_1, μ_2, μ_3 for servers S_1, S_2 and S_3, respectively. Draw the system diagram for server S_2. (2p)

d) Consider a network of three nodes shown in Fig. 1. Nodes 1 and 2 are M/M/1 queuing systems, and node 3 is an M/M/k/k loss system. Packets arrive to node 1 with intensity λ. Packets leaving from node 1 go to node 2 with probability p_1, and those from node 2 go back to enter node 1 with probability p_2. The loss probability at node 3 is p_b. What is the loss probability of an arbitrary packet? (2p)

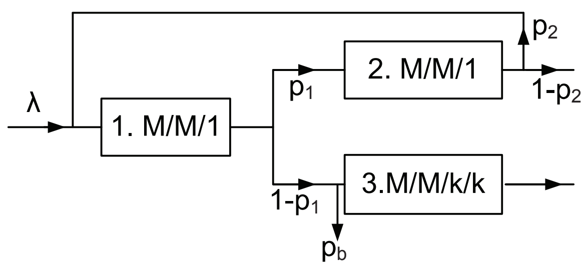

Figure 1: Queuing network for problem 5d