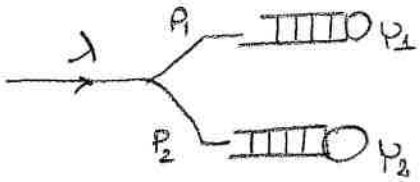
EP2200 Exam 2012 fall

Problem 1



Poisson Split $\rightarrow$ Two independent queues

M/M/1/K , $K = 11$

1) $\lambda_1 = p_1 \lambda = 400$; $\mu_1 = 500$ $\rightarrow$ $\rho_1 = 0{,}8$

2) $\lambda_2 = p_2 \lambda = 600$, $\mu_2 = 1000$ $\rightarrow$ $\rho_2 = 0{,}6$

a)

M/M/1/K formulas

$P(S_1) = \Pr\{\text{arrival at } S_1 \text{ is rejected}\} = \dfrac{(1-\rho_1)\rho_1^{11}}{1-\rho_1^{12}} \approx 0{,}01845$

$P(S_2) = \Pr\{\text{arrival at } S_2 \text{ is rejected}\} = \dfrac{(1-\rho_2)\rho_2^{11}}{1-\rho_2^{12}} \approx 1{,}45 \cdot 10^{-3}$

$P(\text{rejected}) = p_1 \cdot P(S_1) + p_2 \cdot P(S_2) \approx \boxed{0{,}00825}$

- $P(\text{rejected by } S_1 \mid \text{rejected}) = \dfrac{p_1 \cdot P(S_1)}{P(\text{rejected})} \approx \boxed{0{,}8945}$

b) Denote:

- $P(\text{rejected by } S_1 \mid \text{rejected}) = P(S_1 \mid r)$
- $P(S_k \mid S_j) = \Pr\{\text{second pkt rejected by-}k \mid \text{first pkt rejected by-}j\}$

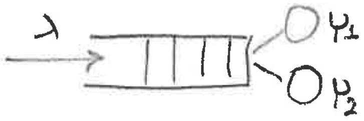
$P(2 \text{ consecutive rejections}) = [p_1 \cdot P(S_1 \mid S_1) + p_2 \cdot P(S_2 \mid S_1)] \cdot P(S_1 \mid r) +$
$+ [p_1 \cdot P(S_1 \mid S_2) + p_2 \cdot P(S_2 \mid S_2)] \cdot P(S_2 \mid r)$

- $P(S_1 \mid S_2) = P(S_1) \dfrac{\lambda}{\lambda + \mu_1}$
- $P(S_2 \mid S_1) = P(S_2) \dfrac{\lambda}{\lambda + \mu_2}$

$\Big\}$ independent queues

- $P(S_1 \mid S_1) = \Pr\{\text{arrival at } S_1 \text{ before departure}\} = \displaystyle\int_0^\infty (\lambda e^{-\lambda t}) \cdot e^{-\mu_1 t} dt = \frac{\lambda}{\lambda + \mu_1}$
- $P(S_2 \mid S_2) = \ldots = \dfrac{\lambda}{\lambda + \mu_2}$

c) Common queue



$P(2 \text{ consecutive blocks}) = \Pr(\text{arrival before departure from any server})$

$$= \int_0^\infty (\lambda e^{-\lambda t}) \cdot e^{-(\mu_1+\mu_2)t}\, dt = \frac{\lambda}{\lambda+\mu_1+\mu_2} =$$

$$= \frac{1000}{2500} = \boxed{0{,}4}$$

d) $\Pr\{W > 10\text{msec}\} = \Pr\{\text{at most 6 departures in } 10\text{msec}\}$

Departure process (from a fully-loaded servers) : Poisson $(\mu_1+\mu_2)$

$$\Rightarrow \Pr(W > 10\text{msec}) = \sum_{k=0}^{6} \frac{((\mu_1+\mu_2)\cdot 10\text{msec})^k}{k!} \cdot e^{-(\mu_1+\mu_2)10\text{msec}} \approx \boxed{9{,}92\cdot 10^{-3}}$$

2. Finite population system

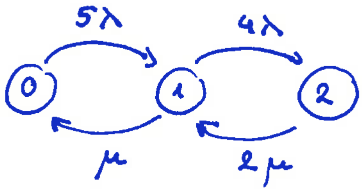
Population: $C = 5$

Servers: $m = 2$

Idle time: $\text{Exp}(\lambda)$ $\lambda = 3/h$ $\bar{x} = 20\text{min} = \frac{1}{3}$ hours

Service time $\text{Exp}(\mu)$ $\mu = 12/h$ $\bar{\tau} = 5\text{min} = \cancel{\frac{1}{12}\text{ hours}}$ $\frac{1}{12}$ hours $\qquad \frac{\lambda}{\mu} = \frac{1}{4}$

a) M/M/2/2/5



$p_0 \cdot 5\lambda = p_1 \cdot \mu \qquad p_1 = 5\frac{\lambda}{\mu}p_0 = \frac{5}{4}p_0$

$p_1 \cdot 4\lambda = p_2 \cdot 2\mu \qquad p_2 = 2 \cdot \frac{\lambda}{\mu}p_1 = 10\left(\frac{\lambda}{\mu}\right)^2 = \frac{10}{16}p_0 = \frac{5}{8}p_0$

$$\left.\begin{aligned} & p_0\left(1 + \frac{5}{4} + \frac{5}{8}\right) = 1 \\ & p_0 \frac{8+10+5}{8} = 1 \end{aligned}\right\} \quad p_0 = \frac{8}{23}, \quad p_1 = \frac{10}{23}, \quad p_2 = \frac{5}{23}$$

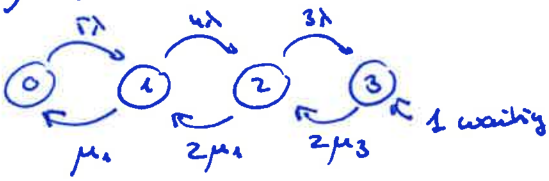
$P(\text{both machines idle}) = p_0 = \frac{8}{23}$

$$\text{utilization} = \frac{p_0 \cdot 0 + p_1 \cdot 1 + p_2 \cdot 2}{2} = \frac{10}{23}$$

b) P(elf finds both machines occupied) = call blocking probability ! =

$$= \frac{3\lambda p_2}{5\lambda p_0 + 4\lambda p_1 + 3\lambda p_2} = \frac{3 \cdot 5}{5 \cdot 8 + 4 \cdot 10 + 3 \cdot 5} = \frac{15}{40+40+15} = \frac{15}{95} = \frac{3}{19} = 0.15 \left(< \frac{5}{23} !\right)$$

c) Now we have two different μ: $\mu_1 = \mu_2 = 12/h$ $\mu_3 = 20/h$ $\qquad \frac{\lambda}{\mu_3} = \frac{3}{20}$



+ equation: $p_2 3\lambda = p_3 \cdot 2\mu_3 \qquad p_3 = p_2 \cdot \frac{3}{2}\frac{\lambda}{\mu_3} = \frac{9}{\cancel{40}_{8}} \cdot \cancel{\frac{5}{8}} p_0 = \frac{9}{64}$

$p_0\left(1 + \frac{5}{4} + \frac{5}{8} + \frac{9}{64}\right) = 1 \qquad p_0 = \frac{64}{193}$

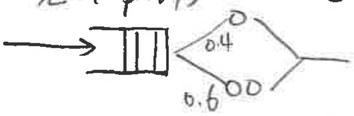
$p_0\left(\frac{64 + 80 + 40 + 9}{64}\right)$

$$P_{block} = \frac{3 \cdot p_3}{5 \cdot p_0 + 4p_1 + 3p_2 + 2 \cdot p_3} = \frac{18}{320 + 320 + 120 + 18} = \frac{18}{778} = 0.023: \text{ very small!}$$

d)
$$T = \frac{N}{\lambda_{eff}} = \frac{1 \cdot p_1 + 2 \cdot p_2 + 3 \cdot p_3}{(5 \cdot p_0 + 4p_1 + 3p_2)\lambda} = 4.64\text{min}$$

Problem 3

$\lambda = 10\,pkt/s$ $\qquad S_1 \sim Exp(u_1)$



$S_2 \sim Erlang(2, u_2)$

$\bar{S}_1 = \frac{1}{u_1} = \frac{1000}{20000} = \frac{1}{20}\,s, \quad u_1 = 20$

$\bar{S}_2 = \frac{2}{u_2} = \frac{2000}{20000} = \frac{1}{10}\,s, \quad u_2 = 20$

a) $\bar{S} = 0.4\,\bar{S}_1 + 0.6\,\bar{S}_2 = \frac{2}{25}\,s$

$\rho = \lambda \bar{S} = \frac{4}{5} = P_{busy}$

△ The calculation is more complicated than we estimated and therefore we have not considered this part for the grading.

b) $M/G/1$, $\bar{w} = \frac{\lambda \overline{S^2}}{2(1-\rho)}$

$\overline{S_1^2} = V[S_1] + \bar{S}_1^2 = \frac{2}{u_1^2} = \frac{1}{200}$

$\overline{S_2^2} = V[S_2] + \bar{S}_2^2 = \frac{6}{u_2^2} = \frac{3}{200}$

$\overline{S^2} = 0.4\,\overline{S_1^2} + 0.6\,\overline{S_2^2} = \frac{11}{1000}$

$\bar{W} = \frac{\lambda \overline{S^2}}{2(1-\rho)} = \frac{11}{40}\,s$

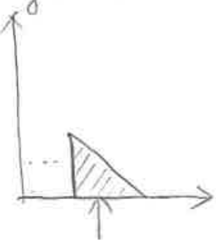
$\bar{T} = \bar{w} + \bar{S} = \frac{71}{200} = 0.355\,s$

$\bar{T}_1 = \bar{W} + \bar{S}_1 = \frac{13}{40} = 0.325\,s$

$\bar{T}_2 = \bar{W} + \bar{S}_2 = \frac{3}{8} = 0.375\,s$

c) $M/G/1$ with vacation

Remaining vacation time



$\overline{R_v|sleep} \sim Uniform(0, 0.1)$

$\overline{R_v|sleep} = \frac{0.1-0}{2} = 0.05\,s$

d) $\bar{w} = \frac{\lambda \overline{S^2}}{2(1-\rho)} + \frac{\overline{V^2}}{2\bar{V}} = \frac{13}{40}$ s

$\bar{T} = \bar{w} + \bar{S} = \frac{81}{200} = 0.405$ s

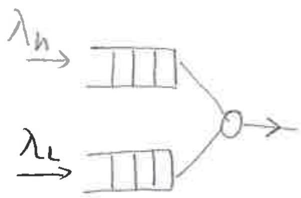
$E_{save} = 10\,J \times (1-\rho) \times 3600 = 7200$ J/h

Increasing V:

1) the mean packet delay increases since the average remaining sleep time increases

2) the mean energy saved in one hour does not change since it depends on the offered load and not depends on V. (At large ρ there will be only few vacation periods.)

Problem 4



$\lambda = 5$ patients/hour

$\lambda_h = 0.2\lambda = 1 \quad p/h$

$\lambda_L = 0.8\lambda = 4 \quad p/h$

$\bar{S}_h = \frac{1}{\mu_h} = 30 \text{ mins} = \frac{1}{2}\ h, \quad \mu_h = 2\ h^{-1}$

$\overline{S_h^2} = V[S_h] + \bar{S}_h^{\,2} = \frac{1}{\mu_h^2} = \frac{1}{2}, \quad \rho_h = \lambda_h \bar{S}_h = \frac{1}{2}$

$\bar{S}_L = \frac{1}{\mu_L} = 5 \text{ mins} = \frac{1}{12}\ h, \quad \mu_L = 12\ h^{-1}$

$\overline{S_L^2} = \frac{1}{\mu_L^2} = \frac{1}{72}, \quad \rho_L = \lambda_L \bar{S}_L = \frac{1}{3}$

a) M/G/1 with preemptive priority

$\bar{R}_h = \frac{1}{2}\lambda_h \overline{S_h^2} = \frac{1}{4}, \qquad \bar{W}_h = \frac{\bar{R}_h}{1-\rho_h} = \frac{1}{2}\ h$

$\bar{R}_L = \frac{1}{2}\lambda_L \overline{S_L^2} + R_h = \frac{5}{18}, \qquad \bar{W}_L = \frac{\bar{R}_L}{(1-\rho_h)(1-\rho_h-\rho_L)} = \frac{10}{3}\ h$

$\bar{w} = 0.2\bar{W}_h + 0.8\bar{W}_L = \frac{83}{30}\ h$

b) high priority: M/M/1 queue

$\bar{T}_h = \bar{W}_h + \bar{S}_h = 1\ h$

$P(W_h > \frac{1}{2}) = \rho_h e^{-(\mu_h - \lambda_h)t}\Big|_{t=\frac{1}{2}} = \frac{1}{2}e^{-\frac{1}{2}} \approx 0.3$

c) P(treatment of LP patient starts imediately)

$=$ P(no hp arrival during $\underbrace{\text{the remaining treatment time of a hp patient}}_{t}$)

$t \sim Exp(\mu_1)$ memoryless

$= \int_0^\infty P(\text{no hp arrival in } t)\, \mu_h e^{-\mu_h t}\, dt$

$= \int_0^\infty e^{-\lambda_h t} \mu_h e^{\mu_h t}\, dt$

$= \frac{\mu_h}{\lambda_h + \mu_h} = \frac{2}{3}$

P4 (contineous)

P(interruption)

$= \int_0^\infty P(\text{at least one hp arrival in } t)\, \mu_L e^{-\mu_L t}\, dt$

$= \int_0^\infty (1 - e^{-\lambda_h t})\, \mu_L e^{-\mu_L t}\, dt$

$= \dfrac{\lambda_h}{\lambda_h + \mu_L} = \dfrac{1}{13}$

d) $P(W > 1h \mid 2 \text{ hp waiting, } 1 \text{ hp servered})$

= P(less than 3 departures from a Poisson Process with rate μ_h)

$= e^{-\mu_h t} + \mu_h t e^{-\mu_h t} + \dfrac{(\mu_h t)^2}{2} e^{-\mu_h t}$

$= 5e^{-2} \approx 0.68$

5.

a) M/M/10/10 , $\lambda = 10\ s^{-1}$ $\mu = 2\ s^{-1}$ $\Rightarrow$ $a = \frac{\lambda}{\mu} = 5$

3

Blocking and non-blocking periods follow each other. $\Rightarrow$

$$P(\text{block}) = \frac{\bar{T}_{\text{block}}}{\bar{T}_{\text{block}} + \bar{T}_{\text{non-block}}} \Rightarrow \underline{\underline{T_{n-b}}} = \frac{1 - P(\text{block})}{P(\text{block})} \cdot T_{\text{block}} = \underline{\underline{2.72\ \text{sec.}}}$$

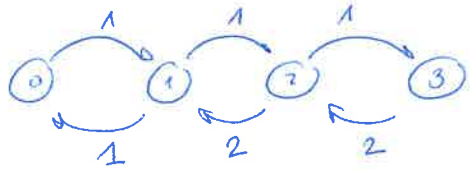
$$P(\text{block}) \overset{\text{Erlang tables}}{=} 0.018$$

$$\bar{T}_{\text{block}} = \{\text{average time to a finished service in state } 10\} = \frac{1}{10\mu} = \frac{1}{20} = 0.05$$

b) M/M/2/3 $\lambda = 10\ s^{-1}$ $\mu = 10\ s^{-1}$ $g = 1$

$\bar{W} = 2$ $\Rightarrow$ service policy does not matter!

3



$p_0 = p_1$

$p_1 = 2p_2$ $\quad p_2 = \frac{1}{2} p_0$

$p_2 = 2p_3$ $\quad p_3 = \frac{1}{4} p_0$

$p_0 (1 + 1 + \frac{1}{2} + \frac{1}{4}) = 1$

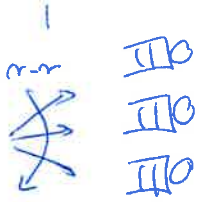
$p_0 = \frac{4}{11}$, $p_1 = \frac{4}{11}$

$p_2 = \frac{2}{11}$, $p_3 = \frac{1}{11}$

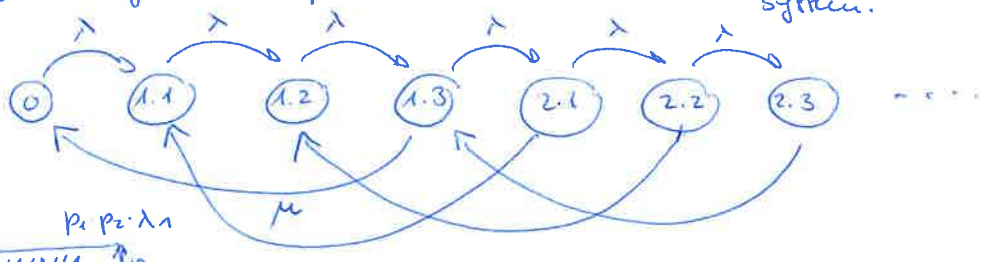
$$\left.\begin{aligned} \lambda_{\text{eff}} &= \lambda(1 - p_b) = \lambda(1 - p_3) = \lambda \cdot \frac{10}{11} \\ N_q &= 1 \cdot p_3 = \frac{1}{11} \end{aligned}\right\} \quad W = \frac{N_q}{\lambda_{\text{eff}}} = \frac{1}{10\lambda} = \frac{1}{100}\ s = \underline{\underline{0.01\ s}}$$

c)

2

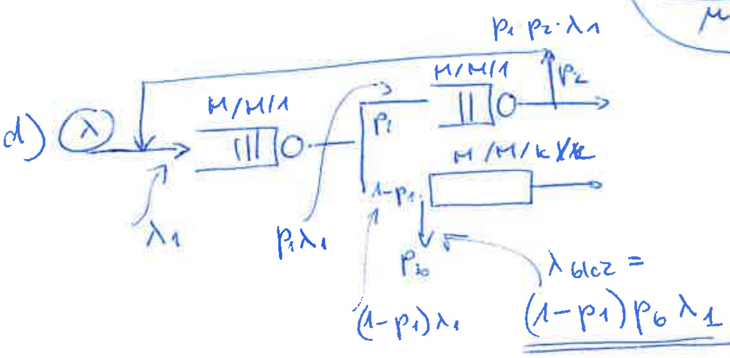


S_2: every 3rd requests arrive to S_2 $\Rightarrow$ $E_3/M/1$ system.



d)

2



We calculate the intensity of packet drops λ_{block}. Then

$$p(\text{block}) = \frac{\lambda_{\text{block}}}{\lambda}$$

$$\lambda + p_1 p_2 \lambda_1 = \lambda_1$$

$$\lambda_1 = \frac{\lambda}{1 - p_1 p_2}$$

$$\boxed{p(\text{block}) = \frac{(1-p_1)p_b}{1 - p_1 p_2}}$$