Positive random variable $X \geq 0$

- **DISCRETE RANDOM VARIABLE**

Probability function: $p_k = P(X = k)$

Properties: $p_k \geq 0$; $\sum_{k=0}^{\infty} p_k = 1$

Probability distribution of X:

$F_X(x_i) = P(X \leq x_i) = \sum_{k=0}^{i} p_k$

Properties: $F_X(x_i) \geq 0$; $F_X(0) = p_0$; $F_X(\infty) = 1$;

$F_X(x_1) \leq F_X(x_2)$ if $x_1 \leq x_2$

Expected (mean) value (first moment) of X:

$E[X] = m = \sum_{i=0}^{\infty} x_i p_i$

Second moment of X: $E[X^2] = \sum_{i=0}^{\infty} x_i^2 p_i$

Variance of X: $Var[X] = E[(X-m)^2] = E[X^2] - m^2$

Squered coefficient of variance: $C^2 = Var[X] / m^2$

- **CONTINUOUS RANDOM VARIABLE**

Probability density function: $f_X(x)$

Properties: $f_X(x) \geq 0$; $\int_0^{\infty} f_X(u)du = 1$

Probability distribution of X:

$F_X(x_i) = P(X \leq x) = \int_0^{x} f_X(u)du$

Properties: $F_X(x_i) \geq 0$; $F_X(0) = 0$; $F_X(\infty) = 1$;

$F_X(x_1) \leq F_X(x_2)$ if $x_1 \leq x_2$

Expected (mean) value (first moment) of X:

$E[X] = m = \int_0^{\infty} x f_X(x)dx$

Second moment of X: $E[X^2] = \int_0^{\infty} x^2 f_X(x)dx$

Variance of X: $Var[X] = E[(X-m)^2] = E[X^2] - m^2$

Squered coefficient of variance: $C^2 = Var[X] / m^2$

Some distributions of X

- DISCRETE **X**

Geometric distributed X:

$p_k = P(X = k) = a^{(k-1)} (1-a); 0<a<1$

$E[X] = 1/a$

Poisson distributed X:

$p_k = P(X = k) = \frac{a^k}{k!} e^{-a}$

$E[X] = a$

- CONTINUOUS **X**

Exponential distributed X:

$f_X(x) = a\, e^{-ax}; 0<a<1$

$E[X] = 1/a$

Erlang$_r$ distributed X:

$f_X(x) = \frac{a^n}{(n-1)!} x^{n-1} e^{-ax}$

$E[X] = n/a$

Exp. distributed random variable *X*

Probability density function: $f_X(t)$	$\mu e^{-\mu t}$
Probability distribution: $F_X(t) = P(X < t)$	$1 - e^{-\mu t}$
Intensity function	μ
Expected (mean) value: $E[X]$	$1/\mu$
Laplace transform of $f_X(t)$	$\frac{\mu}{s + \mu}$

Transforms (moment generating functions)

- **DISCRETE X**

$\mathcal{Z}$ – transform of p_i

$$P(z) = E[z^i] = \sum_{i=0}^{\infty} z^i p_i$$

$$\frac{dP(z)}{dz} = \sum_{i=0}^{\infty} i \cdot z^{i-1} p_i$$

$$\frac{d^2P(z)}{dz^2} = \sum_{i=0}^{\infty} i(1-i) z^{i-2} p_i$$

E[X] = P '(z) for z = 1

E[X²] = P ''(z) + E[X] for z = 1

- **CONTINUOUS X**

$\mathcal{L}$ – transform of $f_X(x)$

$$F^*(s) = E[e^{-sX}] = \int_0^{\infty} e^{-sX} f_X(x)dx$$

$$\frac{dF^*(s)}{ds} = \int_0^{\infty} -x e^{-sx} f(x)dx$$

$$\frac{d^2F^*(s)}{ds^2} = \int_0^{\infty} x^2 e^{-sx} f(x)dx$$

E[X] = - F* '(s) for $s = 0$

E[X²] = F* ''(s) for $s = 0$

Classification of stochastic processes

- Stochastic process SP: $X(t,\omega)$

Random variable X

Discrete X **Continuous X**

  - Discrete-time SP (discrete t): $X(t,\omega)$
  - Continuous-time SP (continuous t): $X(n,\omega)$
- Markov process (MP): a *memoryless* SP
- Markov chain (MC): MP with discrete $\mathbf{X}$
  - Discrete time MC (**DTMC**)
  - Continuous time MC (**CTMC**)
    - Birth-death process (**B-D** process), a special case of CTMC
    - **Poisson process**

Markov process

- Markov process
  - memoryless stochastic process
  - future depends on the present state only
- Continuous-time Markov chains
  - transition intensity matrix–stationary solution
  - global and local balance equations
- Birth-death process
  - transitions only between neighboring states
- Poisson process
  - number of events in a time interval has Poisson distribution
  - time intervals between events has exponential distribution

Kendall's notation

A / B / X / Y / Z

- **A**: arrival process
- **B**: service time
- X: number of servers
- Y: maximum occupancy (not indicated if unlimited buffer)
- Z: service order or other specifications (not indicated if FCFS and no further spec.)
- **A and B** (arrival process and service time) can be:
  - ***M***: Markov (memoryless): exponential distributed time
  - ***D***: deterministic
  - E_r: r exponential distributed steps
  - H_k: hyper exponential with k branches
  - ***G***: general (but known)