SOLUTION 4.7

Detection in a Rayleigh fading channel

We have

$$\begin{aligned} P(e) &= \mathbf{E}[\mathbf{Q}(\sqrt{\mathbf{2|h|^2}\text{SNR}})], \\ &= \int_0^\infty e^{-x} \int_{\sqrt{2x\text{SNR}}}^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt dx = \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_0^{t^2/(2\text{SNR})} e^{-t^2/2} e^{-x} dx dt \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t^2/2} \left(1 - e^{-t^2/(2\text{SNR})}\right) dt = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-t^2(1+1/\text{SNR})/2} dt \\ &= \frac{1}{2}\left(1 - \sqrt{\frac{\text{SNR}}{1+\text{SNR}}}\right), \end{aligned}$$

In the first step we take into account that $|h|$ is a Rayleigh random variable; i.e., it has the density $\frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}}$ and hence its squared magnitude $|h|^2$ is exponentially distributed with density $\frac{1}{\sigma^2} e^{\frac{-x}{\sigma^2}}$, $x \geq 0$. Remember that according to the question assumptions $\sigma = 1$. Moreover, the third step follows from changing the order of integration.

Now, for large SNR, Taylor series expansion yields

$$\sqrt{\frac{\text{SNR}}{1+\text{SNR}}} = 1 - \frac{1}{2\text{SNR}} + \mathbf{O}\left(\frac{\mathbf{1}}{\text{SNR}^\mathbf{2}}\right) \sim \mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}\text{SNR}}$$

which implies

$$P(e) \sim \frac{1}{4\text{SNR}}.$$

SOLUTION 4.8

Average error probability for Log-normal fading

First, we present a short summary of the Stirling approximation. A natural way of approximating a function is using the Taylor expansion. Specially, for a function $f(\theta)$ of a random variable θ having mean μ and variance σ^2, using the Taylor expansion about the mean we have

$$f(\theta) = f(\mu) + (\theta - \mu) f'(\mu) + \frac{1}{2}(\theta - \mu)^2 f''(\mu) + \cdots.$$

By taking expectation

$$\mathbf{E}\{f(\theta)\} \sim f(\mu) + \frac{1}{2} f'(\mu)\sigma^2.$$

However, in Stirling approximation one can start with these differences

$$f(\theta) = f(\mu) + (\theta - \mu)\frac{f(\mu + h) - f(\mu - h)}{2h} + \frac{1}{2}(\theta - \mu)^2 \frac{f(\mu + h) - 2f(\mu) + f(\mu - h)}{h^2} + \cdots,$$

then, taking the expectation we have

$$\mathbf{E}\, f(\theta) \sim f(\mu) + \frac{1}{2}\frac{f(\mu + h) - 2f(\mu) + f(\mu - h)}{h^2}\sigma^2.$$

It has been shown that $h = \sqrt{3}$ yields a good result. So we obtain $\mathbf{Q}(\gamma)$. Given a log-normal random variable z with mean μ_z and variance σ_z^2, we calculate the average probability of error as the average of $\mathbf{Q}(\gamma)$. Namely,

$$\mathbf{E}\{\mathbf{Q}(z)\} \sim \frac{2}{3}\mathbf{Q}(\mu_z) + \frac{1}{6}\mathbf{Q}(\mu_z + \sqrt{3}\sigma_z) + \frac{1}{6}\mathbf{Q}(\mu_z - \sqrt{3}\sigma_z).$$