Wave Response of Ideal Media

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Overview

- Introduction to the concept of a response
- First example: Response of electron gas
  - Changing the speed of light
  - Dispersion
- Polarization of atoms and molecules (brief)
  - Response of crystals
- Medium of forced oscillators
  - Detailed study of the resonance region
    - Hermitian / antihermitian parts of the dielectric tensor
    - Application of the Plemej formula
- Dielectric response for plasmas
  - Magnetoionic theory (anisotropic/gyrotropic)
  - Cold plasmas (Alfven velocity)
  - Warm plasmas (Landau damping)

What do we mean by dielectric response?

- When an electromagnetic wave passes through a media, e.g. air, water, copper, a crystal or a plasma, then:
  - The electromagnetic fields exert a force on the particles of the media
  - The force may then "pull" the particles to induce
    - charge separation ρ ⇨ drive E-field in Poisson's equation

$$\nabla \bullet \mathbf{E}_{media} = \rho_{media} / \varepsilon_0$$

  - $\mathbf{E}$-field is coupled to the $\mathbf{B}$-field through Maxwells equations
    - currents $\mathbf{J}$ ⇨ drive $\mathbf{E}$- & $\mathbf{B}$-fields through Ampere's law

$$\nabla \times \mathbf{B}_{media} = \mu_0 \mathbf{J}_{media} + \frac{1}{c^2} \frac{\partial \mathbf{E}_{media}}{\partial t}$$

  - The fields induced by the media is called the *dielectric response*
  - The total fields are:

$$\mathbf{E} = \mathbf{E}_{external} + \mathbf{E}_{media}$$

$$\mathbf{B} = \mathbf{B}_{external} + \mathbf{B}_{media}$$

See previous lecture for representation in terms of:
- Polarization P
- Magnetization M

Equations for calculating the dielectric response

E- & B-field exerts force on particles in media

$$m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad \textit{solve for } \mathbf{v}\text{ !}$$

The induced motion of charge particles form a current and a charge density

$$\mathbf{J}_{media} = \sum_{species} qn\mathbf{v} \qquad \frac{\partial}{\partial t}\rho_{media} + \nabla \circ \mathbf{J}_{media} = 0$$

(n=particle density)

Current and charge drive the **dielectric response**

$$\nabla \bullet \mathbf{E}_{media} = \rho_{media} / \varepsilon_0$$

$$\nabla \times \mathbf{B}_{media} - \frac{1}{c^2}\frac{\partial \mathbf{E}_{media}}{\partial t} = \mu_0 \mathbf{J}_{media}$$

The respons can be quantified in e.g. the conductivity σ

$$J_i(\mathbf{k},\omega) = \sigma_{ij}(\mathbf{k},\omega) E_j(\mathbf{k},\omega)$$

Response of *electron gas* to oscillating E-field

- The response of a media is driven by the electric and magnetic forces on the particles in the media

> Use Newton's equations, or quantum mechanics, to describe how the charged particles moves and thus the response of the medium

- Example: Consider electron response to electric field oscillations (e.g. high frequency, long wave length waves in a plasma)
  - Align x-axis with the electric field: $\mathbf{E}(t) = \mathbf{e}_x E_x(t)$
  - Electron equation of motion:

$$m\ddot{x}(t) = qE_x(t) \quad \Longrightarrow \quad x(\omega) = -\frac{q}{m\omega^2} E_x(\omega)$$

  - The current driven in the medium (let n be the electron density)

$$J_x(t) \equiv qn\dot{x}(t) \quad \Longrightarrow \quad J_x(\omega) = i\frac{q^2 n}{m\omega} E_x(\omega)$$

  - Thus we have derived the conductivity of this media

$$\sigma(\omega) = i\frac{nq^2}{m\omega}$$

  - Note: $\sigma(\omega)$, i.e. the media is *dispersive*!

Response of *electron gas* to oscillating E-field (2)

- This media is ***isotropic*** (the same response in all directions)
  - **Proof 1**: rotate E-field to align with y-axis or z-axis and repeat calculation
  - **Proof 2**: use argument that the medium have no "intrinsic direction" (like a static the magnetic field or the structure of a crystal), thus the media have to be isotropic
  - Being an isotropic media the components of the conductivity tensor are:

$$\sigma_{ij}(\omega) = \sigma(\omega)\delta_{ij} = i\frac{q^2 n}{m\omega}\delta_{ij} \equiv i\varepsilon_0 \frac{\omega_p^2}{\omega}\delta_{ij}$$

where ω_p is known as the *plasma frequency* $\omega_p^2 \equiv \dfrac{nq^2}{\varepsilon_0 m}$

- Relations to:
  - susceptibility: $$\chi_{ij}(\omega) \equiv \frac{i}{\varepsilon_0\omega}\sigma_{ij}(\omega) = \frac{i\sigma(\omega)}{\varepsilon_0\omega}\delta_{ij} = -\frac{\omega_p^2}{\omega^2}\delta_{ij}$$
  - polarisation response: $$\alpha_{ij}(\omega) \equiv i\omega\sigma_{ij}(\omega) = -\varepsilon_0\omega_p^2\delta_{ij}$$
  - dielectric tensor: $$K_{ij}(\omega) \equiv \delta_{ij} + \chi_{ij}(\omega) = \left(1 - \frac{\omega_p^2}{\omega^2}\right)\delta_{ij}$$

Application of response

- How does the electron response affect the propagation of waves?
  - Consider: high frequency, long wave length waves in a plasma
    - then response tensor from previous page is valid (more details later)
- Split currents into antenna current J_{ant} and the current induced in the media J_{media}. Then Amperes and Faradays equations give:

$$\mathbf{k} \times \mathbf{k} \times \mathbf{E} + \boxed{\frac{\omega^2}{c^2}\mathbf{E} + i\mu_0\omega\mathbf{J}_{media}} = -i\mu_0\omega\mathbf{J}_{ant}$$

Note: total field E driven by both J_{media} and J_{ant}

- Use the conductivity $\sigma = i\varepsilon_0 \frac{\omega_p^2}{\omega}$ of the media:

$$\frac{\omega^2}{c^2}\mathbf{E} + i\mu_0\omega\mathbf{J}_{media} = \frac{\omega^2}{c^2}\mathbf{E} + i\mu_0\omega\sigma\mathbf{E} = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)\mathbf{E}$$

$$\Rightarrow \quad \mathbf{k} \times \mathbf{k} \times \mathbf{E} + \frac{\omega^2}{c_m^2}\mathbf{E} = -i\mu_0\omega\mathbf{J}_{ant}$$

i.e. a wave equation with speed of light: $c_m^2 = c^2\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1}$

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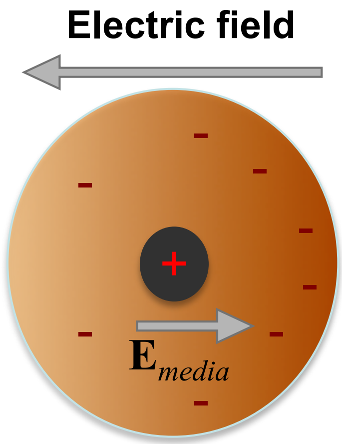
Polarization of atoms and molecules

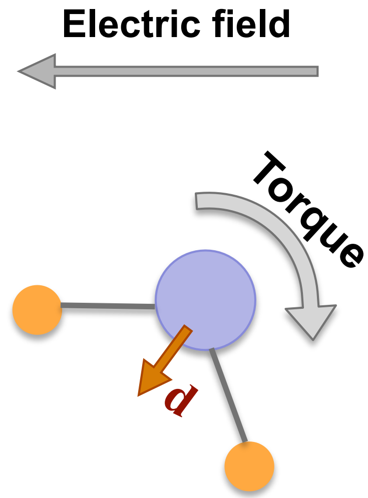
- The polarization of an atom; (require quantum mechanics)
  - The electric field pushes the electrons, inducing a charge separation;
  - Quantum mechanically: perturbs the eigenfunctions (orbitals): $\psi^{(0)} \Rightarrow \psi^{(0)}+\psi^{(1)}$

$$\psi_q^{(1)} = \sum a_{qq'}\psi_{q'}^{(0)} \rightarrow J_{media}^{(1)} \ \& \ \rho_{media}^{(1)}$$

  - The field induced by the media is opposite to the total field

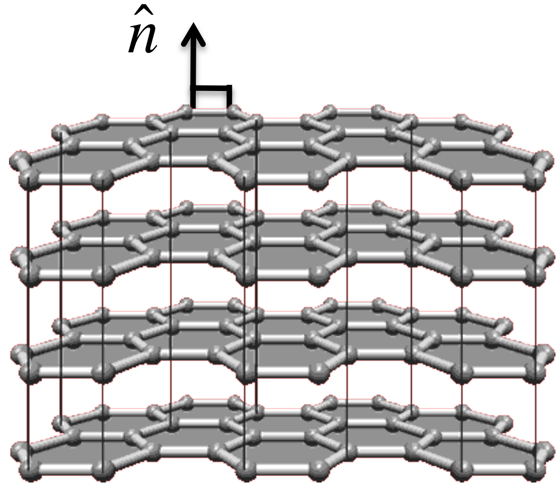


- The polarization of a water molecule
  - Water molecules, dipole moment **d**
  - The electric field induce a *torque* that turns it to reduce the total field
  - Note: the electron eigenfunction in molecule are also perturbed, like in the atom



Uniaxial crystals

- In solids the response, or electron mobility, is determined by the
  - **Metals**: the valence electron give rapid response
  - **Insulators**: electrons orbitals are bound to a single atom or molecule
- Uniaxial crystals: have an optical axis; e.g. the normal $\hat{n}$ to a sheeth structure



Crystal structure of graphite

- Stronger bonds within then between the sheeths
  - Graphite: valence electrons are shared only within a sheeth
  - electron ***mobility*** (response) is different within and perpendicular to the sheeths
  - The crystal is ***anisotropic***
- Let the normal to the crystal be in the z-direction (as in figure)

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_\perp & 0 & 0 \\ 0 & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{bmatrix}$$

Graphite

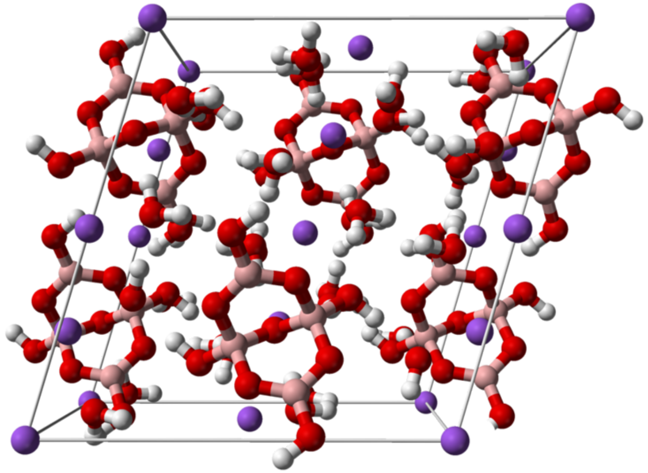
$$\begin{cases} \sigma_\parallel = 2.5 - 5.0 \times 10^{-6} \\ \sigma_\perp = 3 \times 10^{-3} \end{cases}$$

- Example: slight birefrigence in optical fibres can cause *modal dispersion*

Biaxial crystals

- Uniaxial crystals has symmetric plane, in which the electron mobility is constant
- Biaxial crystals have symmetry plane
  - Instead they have different conductivity in all three directions

$$\left[\sigma_{ij}\right] = \begin{bmatrix} \sigma_\alpha & 0 & 0 \\ 0 & \sigma_\beta & 0 \\ 0 & 0 & \sigma_\gamma \end{bmatrix}$$



Biaxial crystal called Borax, $Na_2(B_4O_5)(OH)_4 \cdot 8(H_2O)$

  - When expressed in terms of the dielectric tensor one may introduce three refractive indexes of the media

$$\left[K_{ij}\right] = \left[\delta_{ij} + \frac{i}{\omega\varepsilon_0}\sigma_{ij}\right] = \begin{bmatrix} (n_\alpha)^2 & 0 & 0 \\ 0 & (n_\beta)^2 & 0 \\ 0 & 0 & (n_\gamma)^2 \end{bmatrix}$$

Epsom Salt ($MgSO_4$):
n_j = [1.433, 1.455, 1.461]

These medias are rarely strongly unisotropic

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Reminder: Equations for calculating the dielectric response

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The induced motion of charge particles form a current and a charge density

$$\mathbf{J}_{media} = \sum_{species} qn\mathbf{v} \qquad \frac{\partial}{\partial t}\rho_{media} + \nabla \circ \mathbf{J}_{media} = 0$$

(n=particle density)

Current and charge drive the **dielectric response**

$$\nabla \bullet \mathbf{E}_{media} = \rho_{media} / \varepsilon_0$$

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$$J_i(\mathbf{k},\omega) = \sigma_{ij}(\mathbf{k},\omega) E_j(\mathbf{k},\omega)$$

Medium of forced oscillators

- Consider a medium consisting of charged particles with
  - charge q , mass m , density n
- Let the particles position x follow the equation of a forced oscillator
  - i.e. the media has an eigenfrequency Ω and a damping rate Γ
    - damping could be due to collisions (resistivity) and the eigenfrequency could be due to magnetization an acustic eigenfrequency of a crystal

$$\ddot{x}(t) + \Gamma \dot{x}(t) + \Omega^2 x(t) = \frac{q}{m} E_x(t) \quad \Rightarrow \quad x(\omega) = \frac{q/m}{\Omega^2 - \omega^2 - i\Gamma\omega} E_x(\omega)$$

- The current is then

$$J(\omega) = qn\left[-i\omega x(\omega)\right] = -\frac{i\omega n q^2/m}{\Omega^2 - \omega^2 - i\Gamma\omega} E_x(\omega) \equiv \sigma E_x(\omega)$$

- Thus the dielectric tensor reads

$$K_{ij} = \delta_{ij} + \frac{i}{\varepsilon_0 \omega}\sigma_{ij} = \left(1 + \frac{\omega_p^2}{\Omega^2 - \omega^2 - i\Gamma\omega}\right)\delta_{ij} \quad , \quad \text{where} \quad \omega_p^2 \equiv \frac{nq^2}{\varepsilon_0 m}$$

  - again ω_p is the plasma frequency

Medium of forced oscillators (2)

- Isotropic dielectric tensors K_{ij} can be replaced by a scalar K, consider e.g. the inner product $K_{ij}E_j = K\delta_{ij}E_j = KE_i$
- For the medium of harmonic oscillators

$$K = 1 + \frac{\omega_p^2}{\Omega^2 - \omega^2 - i\Gamma\omega}$$

- In the high frequency limit where $\omega >> \Omega$ and $\omega >> \Gamma$, then

$$K = 1 - \frac{\omega_p^2}{\omega^2} + \ldots$$

  - this is the response of the electron gas!
- At low frequency $\omega << \Omega$ and $\Omega \sim \Gamma$, then

$$K = 1 + \frac{\omega_p^2}{\Omega^2}$$

  - here the medium is no longer dispersive (independent of ω)

Medium of forced oscillators (3)

- The medium is the most dispersive when the frequency is near the characteristic frequency of the medium $\omega \sim \Omega$,
  - first rewrite the denominator

$$D \equiv \Omega^2 - \omega^2 - i\Gamma\omega =$$

$$= \Omega^2 - (\omega + i\Gamma/2)^2 - \Gamma^2/4$$

$$= (\Omega - \omega - i\Gamma/2)(\Omega + \omega + i\Gamma/2) - \Gamma^2/4$$

  - assume here the damping rate to be small $\omega >> \Gamma$ such that the last last term is negligible
  - Next use the relation: $\frac{1}{(a-b)(a+b)} = \frac{1}{2b}\left(\frac{1}{a-b} - \frac{1}{a+b}\right)$
  - The dielectric constant is then

$$K \approx 1 - \frac{\omega_p^2}{(\omega + i\Gamma/2 - \Omega)(\omega + i\Gamma/2 + \Omega)}$$

$$= 1 - \frac{\omega_p^2}{2\Omega}\left[\frac{1}{\omega + i\Gamma/2 - \Omega} - \frac{1}{\omega + i\Gamma/2 + \Omega}\right]$$

Medium of forced oscillators (4)

- Next we shall use the condition that we are close to resonance; i.e. the frequency is near the characteristic frequency $\omega \sim \Omega$:

$$|\omega - \Omega| << |\omega + \Omega| \quad \Rightarrow \quad \left|\frac{1}{\omega - \Omega + i\Gamma/2}\right| >> \left|\frac{1}{\omega + \Omega + i\Gamma/2}\right|$$

- The dielectric constant then reads

$$K \approx 1 - \frac{\omega_p^2}{2\Omega}\frac{1}{(\omega - \Omega + i\Gamma/2)} = 1 - \frac{\omega_p^2}{2\Omega}\frac{(\omega - \Omega - i\Gamma/2)}{\left[(\omega - \Omega)^2 + \Gamma^2/4\right]}$$

$$K^H \equiv \Re\{K\} = 1 - \frac{\omega_p^2}{2\Omega}\frac{\omega - \Omega}{\left[(\omega - \Omega)^2 + \Gamma^2/4\right]}$$

Hermitian: wave propagation (reactive response)

$$K^A \equiv \Im\{K\} = \frac{\omega_p^2}{\Omega}\frac{\Gamma}{\left[(\omega - \Omega)^2 + \Gamma^2/4\right]}$$

Antihermitian: wave absorption (resistive response)

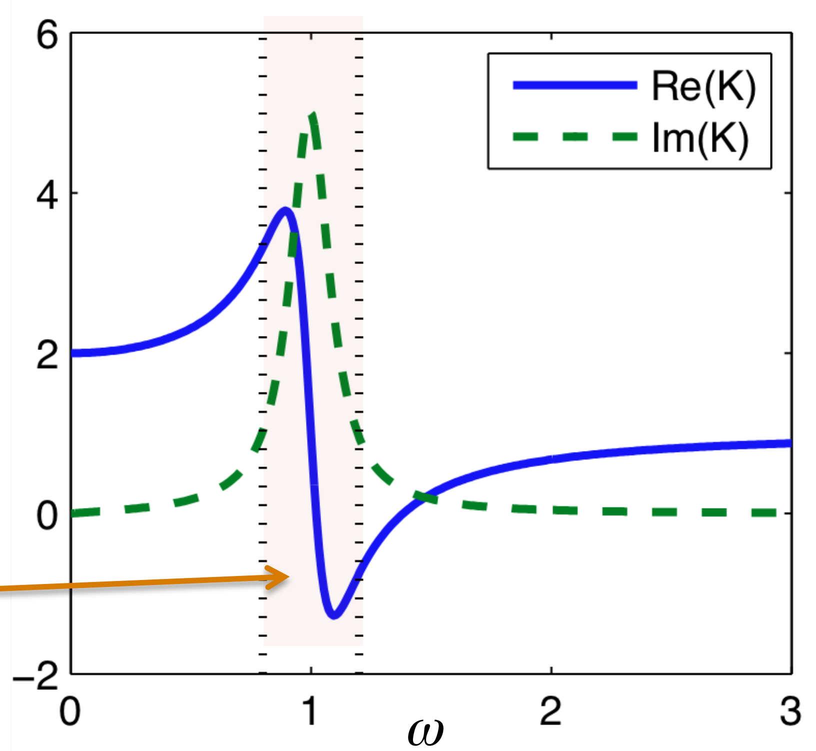
Medium of forced oscillators (5)

- Antihermitian part comes from $i\Gamma/2$ in

$$K \approx 1 - \frac{\omega_p^2}{2\Omega} \frac{1}{(\omega - \Omega + i\Gamma/2)}$$

  – which is most important if $|\Gamma/2| \sim |\omega - \Omega|$ (for $|\Gamma/2| << |\omega - \Omega|$ then $K^A << K^H$)

  – Thus, the dissipation occur mainly where $|\Gamma| > |\omega - \Omega|$



- Summary:
  – Low frequency: not dispersive
  – Resonant region: strong damping in ***thin layer*** $|\Gamma| > |\omega - \Omega|$
  – High frequency: response decay with frequency, $\chi \sim K - 1 \sim \omega^{-2}$ like an electron gas.

Medium of forced oscillators (6)

- What happens in the limit when the damping Γ goes to zero?
- Again assume $\omega \sim \Omega$ then

$$K \approx 1 - \frac{\omega_p^2}{2\Omega} \frac{1}{(\omega - \Omega + i\Gamma/2)}$$

- The limit where Γ goes to zero can be rewritten using the Plemej formula

$$\lim_{\Gamma \to 0} K \approx \lim_{\Gamma \to 0} \left(1 - \frac{\omega_p^2}{2\Omega} \frac{1}{(\omega - \Omega + i\Gamma/2)} \right) = 1 - \frac{\omega_p^2}{2\Omega} \frac{1}{(\omega - \Omega + i0)} =$$

$$= 1 - \frac{\omega_p^2}{2\Omega} \left[\wp \frac{1}{\omega - \Omega} - i\pi\delta(\omega - \Omega) \right]$$

- Thus when the damping goes to zero there is still an imaginary part that will cause wave damping!

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Dielectric response for plasmas

- Plasma ~ ionized gas
  - a plasma is a collection of ions and electrons
  - **Charge neutrality**: highly conducting; insteady state charge density is zero
  - plasmas tend to follow classical mechanics (not quantum mechanics), i.e. Newton/Einsteins equations of motion combined with Maxwell's equations
- First example is the Magnetoionic theory:
  - ***Assume***: ions are static; unperturbed by the wave field (no response)
  - ***Assume***: electrons are cold; they are initially static, but move in the presence of the wave field
  - the plasma has a static magnetic field; align the coordinate system: $\mathbf{B}_0 = B_0\mathbf{e}_z$
  - align also y-axis with the $\mathbf{k}$ vector: $\mathbf{k} = k_y\mathbf{e}_y + k_z\mathbf{e}_z$
  - the respons of the electrons is then given by Newtons equation

$$\dot{\mathbf{v}} = \frac{q}{m}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

  - Next: add a friction with the ions (a force $-m\nu_f\mathbf{v}$) and use $\mathbf{v} = \dot{\mathbf{r}}$

$$\ddot{\mathbf{r}} = \frac{q}{m}\left(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}\right) - \nu_f\dot{\mathbf{r}}$$

Dielectric response for plasmas (2)

- Note that the magnetic field has two components a wave component and a static component

$$\mathbf{B} = \mathbf{B}_{wave} + \mathbf{B}_0$$

  – Thus the Lorentz force is non-linear: $m\dot{\mathbf{r}} \times \left(\mathbf{B}_{wave} + \mathbf{B}_0\right)$

- Assuming that the wave amplitude is small, then we can neglect $\mathbf{B}_{wave}$

$$\ddot{\mathbf{r}} - \dot{\mathbf{r}} \times \mathbf{e}_z \frac{q}{m} B_0 + \nu_f \dot{\mathbf{r}} = \frac{q}{m}\mathbf{E}$$

  – here we can identify the cyclotron frequency $\Omega = qB_0/m$

- Fourier transform: $-\omega^2 \mathbf{r} + i\omega \mathbf{r} \times \mathbf{e}_z \Omega - i\omega\nu_f \mathbf{r} = \frac{q}{m}\mathbf{E}$

$$-\omega^2 r_i + i\omega\varepsilon_{ijk} r_j \delta_{3k}\Omega - i\omega\nu_f r_i = \frac{q}{m} E_i$$

Note: $\mathbf{e}_z = \mathbf{e}_3 = \delta_{3k}\,\mathbf{e}_k$

$$\underbrace{\left[\left(\omega + i\nu_f\right)\delta_{ij} - i\varepsilon_{ij3}\Omega\right]}_{\text{Matrix in the indexes } i,\, j} r_j = -\frac{q}{m\omega} E_i$$

Matrix in the indexes i, j

Dielectric response for plasmas (3)

- Write equation as a matrix equations:

$$\left[(\omega + i\nu_f)\delta_{ij} - i\varepsilon_{ij3}\Omega\right]\left[r_j\right] = \begin{bmatrix} \omega + i\nu_f & -i\Omega & 0 \\ i\Omega & \omega + i\nu_f & 0 \\ 0 & 0 & \omega + i\nu_f \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \frac{q}{m\omega}\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

- Inverting the matrix

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \frac{q}{m\omega}\begin{bmatrix} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & 0 \\ 0 & 0 & M_{33} \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

$$\begin{cases} M_{11} = M_{22} = \dfrac{\omega + i\nu_f}{(\omega + i\nu_f)^2 - \Omega^2} \\ M_{12} = -M_{21} = \dfrac{i\Omega}{(\omega + i\nu_f)^2 - \Omega^2} \\ M_{33} = \dfrac{1}{\omega + i\nu_f} \end{cases}$$

- The current is then

$$\mathbf{j} = nq(-i\omega\mathbf{r}) = -i\varepsilon_0\omega_p^2\begin{bmatrix} M_{11} & M_{12} & 0 \\ M_{21} & M_{22} & 0 \\ 0 & 0 & M_{33} \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

Dielectric response for plasmas (4)

- The dielectric tensor in the magnetoionic theory then reads:

$$[K_{ij}] = \left[\delta_{ij} + \frac{i}{\varepsilon_0\omega}\sigma_{ij}\right] = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix}$$

$$\begin{cases} S = 1 - \dfrac{\omega_p^2}{\omega}\dfrac{\omega + i\nu_f}{(\omega + i\nu_f)^2 - \Omega^2} \\ D = -\dfrac{\omega_p^2}{\omega}\dfrac{\Omega}{(\omega + i\nu_f)^2 - \Omega^2} \\ P = 1 - \dfrac{\omega_p^2}{\omega(\omega + i\nu_f)} \end{cases}$$

or

$$K_{ij} = S\left(\delta_{ij} - b_i b_j\right) + P b_i b_j - iD\varepsilon_{ijk} b_k$$

where b_k are the components of the unit vector parallel to the magnetic field

- This dielectric response tensor is:
  - ***Anisotropic***; response is different for $\mathbf{E}$ in the x, y, or z direciton.
  - ***Gyrotropic***: the off-diagonal terms (involving D) are perpendicular to a characteristic direction of the media

Hermitian part of the dielectric tensor

$$\mathbf{K}^H = \frac{1}{2}\left(\mathbf{K}+\mathbf{K}^{T*}\right) = \frac{1}{2}\left(\begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix} + \begin{bmatrix} S^* & (-iD)^* & 0 \\ (iD)^* & S^* & 0 \\ 0 & 0 & P^* \end{bmatrix}^T\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix} + \begin{bmatrix} S^* & (iD)^* & 0 \\ (-iD)^* & S^* & 0 \\ 0 & 0 & P^* \end{bmatrix}\right) =$$

Transpose make $-iD^*$ and iD^* change place

$$= \frac{1}{2}\left(\begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix} + \begin{bmatrix} S^* & -i(D^*) & 0 \\ i(D^*) & S^* & 0 \\ 0 & 0 & P^* \end{bmatrix}\right) =$$

$$= \frac{1}{2}\begin{bmatrix} S+S^* & -i(D+D^*) & 0 \\ i(D+D^*) & S+S^* & 0 \\ 0 & 0 & P+P^* \end{bmatrix} = \begin{bmatrix} \Re\{S\} & -i\Re\{D\} & 0 \\ i\Re\{D\} & \Re\{S\} & 0 \\ 0 & 0 & \Re\{P\} \end{bmatrix}$$

Antihermitian part of the dielectric tensor

$$\mathbf{K}^{A} = \frac{1}{2}\left(\mathbf{K} - \mathbf{K}^{T*}\right) =$$

$$= \frac{1}{2}\left(\begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix} - \begin{bmatrix} S^{*} & (-iD)^{*} & 0 \\ (iD)^{*} & S^{*} & 0 \\ 0 & 0 & P^{*} \end{bmatrix}^{T}\right) =$$

$$= i\begin{bmatrix} \Im\{S\} & -i\Im\{D\} & 0 \\ i\Im\{D\} & \Im\{S\} & 0 \\ 0 & 0 & \Im\{P\} \end{bmatrix}$$

Cold plasma dielectric response

- A commonly used representation of the plasma is the ***cold plasma***
  - Here ions and electrons are in a stationary equilibrium, and move only in the presence of a wave field
  - Usually the friction between ions and electrons are neglected
  - Each species is then described by the
    - charge q^ν
    - mass m^ν
    - position $\mathbf{r}^\nu$ (or velocity $\mathbf{v}^\nu$)
  - where $\nu = i$ represent the ions and $\nu = e$ represent the electrons
    - NOTE: ν is not a tensor index!
- The linearlised equation of motion for species ν ($\mathbf{B}_0 = B_0\,\mathbf{e}_z$) :

$$m^\nu \ddot{\mathbf{r}}^\nu - q^\nu \dot{\mathbf{r}}^\nu \times \mathbf{B}_0 = q^\nu \mathbf{E}$$

$$\ddot{\mathbf{r}}^\nu - \dot{\mathbf{r}}^\nu \times \mathbf{e}_z \Omega^\nu = \frac{q^\nu}{m^\nu}\mathbf{E}$$

  - where $\Omega^\nu = q^\nu B_0 / m^\nu$
  - this equation is solved like in the magnetoionic theory

Cold plasma dielectric response (2)

- The solution of the equation of motion for species ν is

$$\begin{bmatrix} r_1^\nu \\ r_2^\nu \\ r_3^\nu \end{bmatrix} = \frac{q^\nu}{m^\nu \omega} \begin{bmatrix} M_{11}^{\ \nu} & M_{12}^{\ \nu} & 0 \\ M_{21}^{\ \nu} & M_{22}^{\ \nu} & 0 \\ 0 & 0 & M_{33}^{\ \nu} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

$$\begin{cases} M_{11}^{\ \nu} = M_{22}^{\ \nu} = \dfrac{\omega}{\omega^2 - \Omega^{\nu\,2}} \\ M_{12}^{\ \nu} = -M_{21}^{\ \nu} = \dfrac{i\Omega^\nu}{\omega^2 - \Omega^{\nu\,2}} \\ M_{33}^{\ \nu} = \dfrac{1}{\omega} \end{cases}$$

- With many species the current is a sum over the all species:

$$\mathbf{j} = \sum_\nu \mathbf{j}^\nu = \sum_\nu n^\nu q^\nu (-i\omega \mathbf{r}^\nu) = \sum_\nu -i\varepsilon_0 \omega_{p\nu}^2 \begin{bmatrix} M_{11}^{\ \nu} & M_{12}^{\ \nu} & 0 \\ M_{21}^{\ \nu} & M_{22}^{\ \nu} & 0 \\ 0 & 0 & M_{33}^{\ \nu} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

  – thus also the conductivity is a sum over species:

$$\sigma = \sum_\nu -i\varepsilon_0 \omega_{p\nu}^2 \begin{bmatrix} M_{11}^{\ \nu} & M_{12}^{\ \nu} & 0 \\ M_{21}^{\ \nu} & M_{22}^{\ \nu} & 0 \\ 0 & 0 & M_{33}^{\ \nu} \end{bmatrix}$$

Cold plasma dielectric response (3)

- The dielectric tensor for the cold plasma reads

$$[K_{ij}] = \left[\delta_{ij} + \frac{i}{\varepsilon_0 \omega}\sigma_{ij}\right] = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix}$$

$$\begin{cases} S = 1 - \sum_\nu \frac{\omega_{p\nu}^2}{\omega^2 - \Omega^{\nu 2}} \\ D = -\sum_\nu \frac{\omega_{p\nu}^2}{\omega} \frac{\Omega^\nu}{\omega^2 - \Omega^{\nu 2}} \\ P = 1 - \sum_\nu \frac{\omega_{p\nu}^2}{\omega^2} \end{cases}$$

- Low frequency limit $\omega << \Omega^\nu$, $\omega_{p\nu}$

$$S = \ldots = 1 + \frac{c^2}{V_A^2} \approx \frac{c^2}{V_A^2}$$

Alfven velocity

$$D = \ldots \approx 0$$

  – i.e. non-dispersive in S !

- Low frequency tensor:
  – compare: uniaxial crystal
  – describes Alfven wave and plasma oscillations (*see next lecture*)

$$[K_{ij}] = \begin{bmatrix} c^2/V_A^2 & 0 & 0 \\ 0 & c^2/V_A^2 & 0 \\ 0 & 0 & 1 - \sum_\nu \omega_{p\nu}^2/\omega^2 \end{bmatrix}$$

Cold plasma dielectric response (4)

- High frequency limit $\omega >> \Omega^{\nu}$, $\omega_{p\nu}$

$$S = 1 - \sum_{\nu} \frac{\omega_{p\nu}^2}{\omega^2} = P$$

$$D = -\sum_{\nu} \frac{\omega_{p\nu}^2 \Omega^{\nu}}{\omega^3} \sim O(\omega^{-3})$$

$$K_{ij} \approx \left(1 - \sum_{\nu} \frac{\omega_{p\nu}^2}{\omega^2}\right)\delta_{ij} + O(\omega^{-3})$$

Like an electron gas!

Response of a warm plasma

- In a warm plasma (like in a gas) the particles move with an unordered thermal motion;
  - the velocity can be considered random (as in statistical mechanics)
- The vector space of velocity vectors $\mathbf{v}=(v_1, v_2, v_3)$ is call velocity space

Definition:
The distribution function $f(\mathbf{v})$ such that $f(\mathbf{v})dv_1 dv_2 dv_3$ is the probability of finding a particle with velocities in

(v_1 , v_1+dv_1) x (v_2 , v_2+dv_2) x (v_3 , v_3+dv_3).

- Relation to the density n and the average fluid velocity $\langle\mathbf{v}\rangle$ by :

$$\begin{cases} n = \int f(v) d^3v \\ n\langle\mathbf{v}\rangle = \int \mathbf{v} f(v) d^3v \end{cases}$$

- Thus, for an ensamble of species ν (e.g. ion and electron)

$$\mathbf{J} = \sum_\nu q^\nu n^\nu \langle\mathbf{v}\rangle^\nu = \sum_\nu q^\nu \int \mathbf{v} f^\nu(v) d^3v$$

Response of a warm plasma

- In absence of a wave field, assume the velocities of the particles of species ν are distributed in a Maxwellian distribution function with temperature T^{ν} and density n^{ν}

$$f^{M\nu}(v) = \frac{n^{\nu}}{(\sqrt{2\pi} m^{\nu} V^{\nu})^3} \exp\left[-\frac{v^2}{2V^{\nu 2}} \right]$$

  - where V^{ν} is the thermal velocity; $T^{\nu} = m^{\nu} (V^{\nu})^2/2$

- When subject to a wave field, the equation of motion reads

$$m^{\nu} \dot{v}_i(t, \mathbf{r}, \mathbf{v}) = q^{\nu} \left[E_i + \varepsilon_{ijk} v_j B_k \right]$$

- The distribution then evolves according to the Vlasov equation (continuity equation in real and velocity space)

$$\left\{ \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} + \dot{v}_i(t, \mathbf{r}, \mathbf{v}) \frac{\partial}{\partial v_i} \right\} f^{\nu}(t, \mathbf{r}, \mathbf{v}) = 0$$

$$\left\{ \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} + \frac{q^{\nu}}{m^{\nu}} \left[E_i(t, \mathbf{r}) + \varepsilon_{ijk} v_j B_k(t, \mathbf{r}) \right] \frac{\partial}{\partial v_i} \right\} f^{\nu}(t, \mathbf{r}, \mathbf{v}) = 0$$

- Note that the wave field perturbs both E, B and f, thus this equations is non-linear in the perturbation!

Response of a warm plasma (2)

- Separate unperturbed and perturbed quantities

$$\begin{cases} f(t,r,v) = f^{M\nu}(v) + f^{1\nu}(t,r,v) \\ \mathbf{E}(t,r,v) = 0 + \mathbf{E}^1(t,r,v) \\ \mathbf{B}(t,r,v) = 0 + \mathbf{B}^1(t,r,v) \end{cases}$$

- Use Faraday's law to write: $\mathbf{B}^1 = \mathbf{k} \times \mathbf{E}^1 / w$

$$\left\{ \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} + \frac{q^\nu}{m^\nu}\left[E_i^1 + \varepsilon_{ijk} v_j B_k^1\right]\frac{\partial}{\partial v_i} \right\} f^{1\nu}(t,r,v) = -\frac{q^\nu}{m^\nu}\left[E_i^1 + \varepsilon_{ijk} v_j B_k^1\right]\frac{\partial}{\partial v_i} f^{M\nu}(v)$$

Non-linear terms

- Linearlised equations

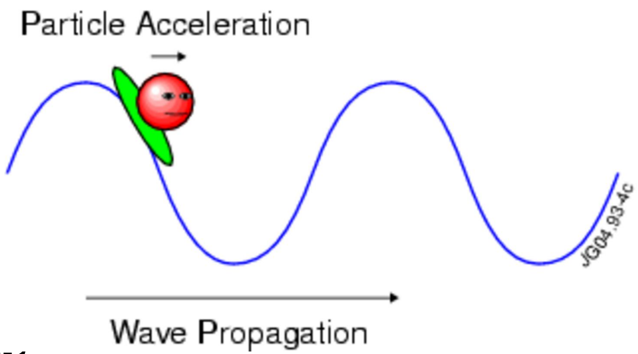
$$\left\{ \frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} \right\} f^{1\nu}(t,r,v) = -\frac{q^\nu}{m^\nu}\left[E_i^1 + \varepsilon_{ijk}\varepsilon_{knm} v_j \frac{k_n}{\omega} E_m^1\right]\frac{\partial}{\partial v_i} f^{M\nu}(v)$$

- Fourier transform $f^{1\nu}(\omega,k,v) = \dfrac{-i}{\omega - \mathbf{k}\bullet\mathbf{v}} \dfrac{q^\nu}{m^\nu}\left[\left(1 - \dfrac{\mathbf{k}\bullet\mathbf{v}}{\omega}\right)\delta_{im} + \dfrac{v_m k_i}{\omega}\right] E_m^1 \dfrac{\partial}{\partial v_i} f^{M\nu}(v)$

Resonance when particles travel
at phase velocity of the wave!

Landau-resonance

- The resonance in the solution to the linearised Vlasov equation is related to a damping $f^{1v}(\omega,k,v) = \frac{i}{\omega - \mathbf{k}\bullet\mathbf{v}}\dots$
  - This was first realised by Landau



- What is the physics of this resonance?
  - Consider a plane wave $E \sim \exp(i\mathbf{k}\bullet\mathbf{x} - i\omega t)$
  - Let a particle travel with the constant velocity $\mathbf{x} = \mathbf{v}t$

$$E \sim \exp(i\mathbf{k}\bullet\mathbf{v}t - i\omega t) = \exp(i[\mathbf{k}\bullet\mathbf{v} - \omega]t) = \exp(-i\omega' t)$$

  - Thus, the particles will *see* a field oscillating with the frequency ω'
    - ω' is the Doppler shifted velocity!
  - The resonance condition $\omega - \mathbf{k}\bullet\mathbf{v} = 0$,
    - i.e. the Doppler shifted frequency is zero
    - i.e. when the travel with the same speed as the wave
    - i.e. the E-field will *accelerate the particle forever* – the wave is damped!
  - Note: we have linearised the equations, thus we assume that change in particle velocity is small no matter now long the acceleration time!
    - in reality non-linear effects come in and then the damping remains only if the dissipation (Γ) is more important than non-linearity

Response of a warm plasma (3)

- The current is now obtained from the integral over velocity space

$$j_n(\omega,k) = \sum_{\nu} q^{\nu} \int v_n f^{1\nu}(\omega,k,v)\,d^3v =$$

$$= \left\{ -i\varepsilon_0 \sum_{\nu} \omega_{p\nu}{}^2 \int \left[\delta_{im} + \frac{v_m k_i}{\omega - \mathbf{k}\bullet\mathbf{v} + i0} \right] \frac{v_n v_i}{n^{\nu} V^{\nu}} f^{M\nu}(v)\,d^3v \right\} E_m{}^1$$

Add a weak dissipation to allow for use of Plemej formula

The conductivity tensor!

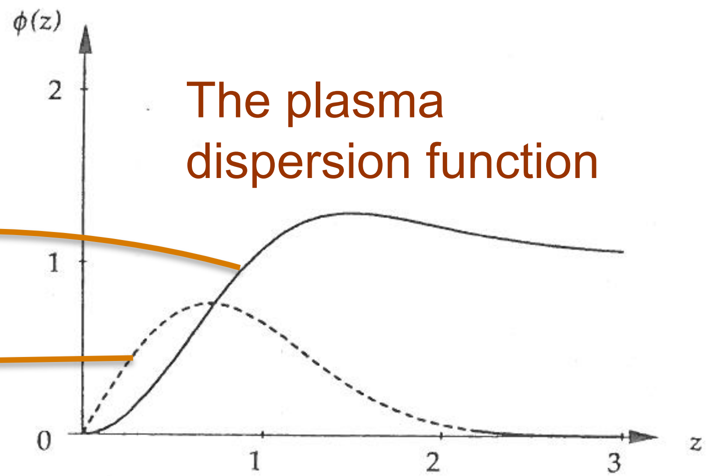
- After some algebra it is possible to rewrite the dielectric tensor as

$$K_{ij} = K^L \kappa_i \kappa_i + K^L\left(\delta_{ij} - \kappa_i \kappa_i\right)$$

$$K^L = 1 + \sum_{\nu} \left(\frac{\omega_{p\nu}}{kV^{\nu}}\right)^2 \left[1 - \phi(y_\nu) + i\sqrt{\pi}\, y_\nu \exp(-y_\nu{}^2)\right]$$

$$K^T = 1 + \sum_{\nu} \left(\frac{\omega_{p\nu}}{\omega}\right)^2 \left[\phi(y_\nu) - i\sqrt{\pi}\, y_\nu \exp(-y_\nu{}^2)\right]$$

$$y_\nu = \omega / \sqrt{2} k V^{\nu} \quad , \quad \phi(z) = 2z e^{-z^2} \int_0^z e^{-t^2}\,dt$$



The plasma dispersion function

Fig. 10.1 The real part (solid curve) and the imaginary part (dotted curve) of the plasma dispersion function (10.27).

Damping in warm plasma

- Consider longitudinal waves
  - the damping is then proportional to (see later lectures for details)

$$\Im\{K^L\} = \sum_\nu \left(\frac{\omega_{p\nu}}{kV^\nu}\right)^2 \sqrt{\pi}\, y_\nu \exp(-y_\nu{}^2)$$

- This function has a maximum when $y_\nu \sim 0.7$, or $\omega/k \sim V_{th}$, i.e. when the phase velocity of the wave is roughly equal to the thermal velocity
  - this the when the Landau resonance is most effective

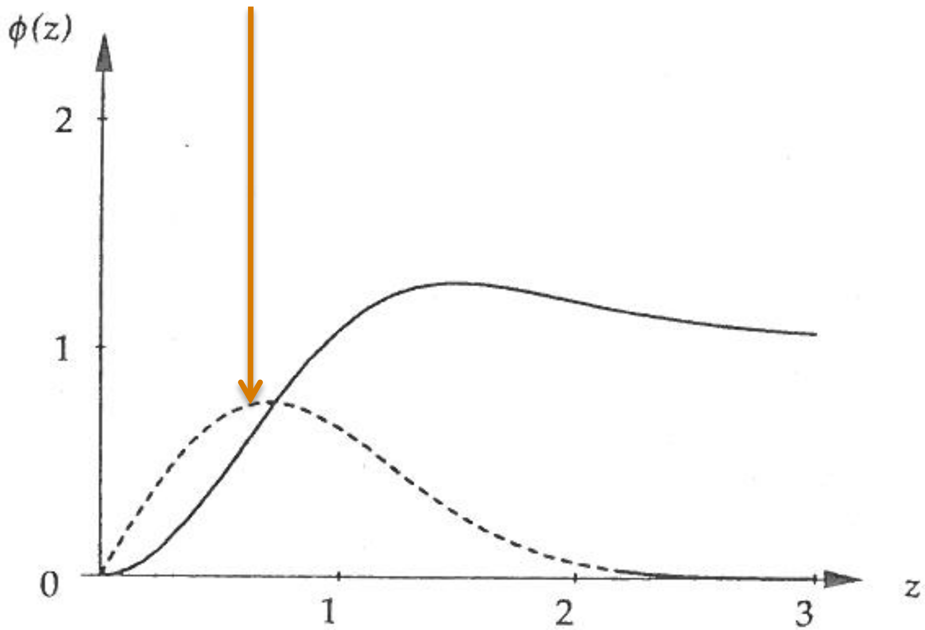


Fig. 10.1 The real part (solid curve) and the imaginary part (dotted curve) of the plasma dispersion function (10.27).