Principles of Wireless Sensor Networks

https://www.kth.se/social/course/EL2745/
Lecture 8

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Course content

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- Part 2
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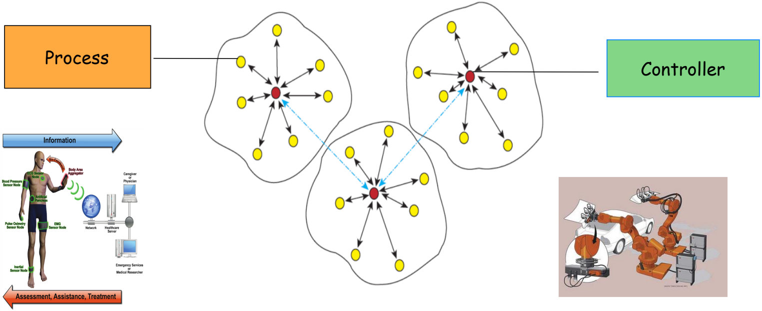
Today's Lecture



Today we study how to perform estimation from noisy measurements of the sensors

Motivation

- plays a central role in many networked applications
- accurately predicts the **parameters** of a **phenomenon**
- **communication:** position, navigation
- **monitoring:** pollutant, earthquake magnitude
- **surveillance:** crowd density, attitude

Today's Learning Goals

- overview on some of the **fundamental aspects** of distributed estimation over networks
  - star topology
  - general topology
  - LMMSE estimate
  - static sensor fusion
- advanced topics
  - sequential measurements from one sensor
  - sequential measurements from many sensors (dynamic sensor fusion)
  - dynamic sensor fusion, distributed Kalman filtering
  - static sensor fusion with limited communication range

Outline

- **Star Topology**
- **General Topology**
- **One Sensor Case**
- **Combining Estimators from Many Sensors (Star Topology)**
- **Sequential Measurements from One Sensor**
- **Sequential Measurements from Many Sensors (Star Topology)**
- **Combining Estimators from Many Sensors (Arbitrary Topology)**

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  - Model of the measurements for one sensor
  - Model of the Estimator
  - Mean Squared Error (MSE) to Chose $\mathbb{L}$
  - LMMSE Estimate
- **Combining Estimators from Many Sensors (Star Topology)**
  - Static Sensor Fusion
- **Sequential Measurements from One Sensor**
- **Sequential Measurements from Many Sensors (Star Topology)**
  - Dynamic Sensor Fusion, Centralized Setup
  - Dynamic Sensor Fusion, Centralized Setup (Drawbacks)
  - Dynamic Sensor Fusion, Distributed Kalman Filtering
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  - Static Sensor Fusion with Limited Communication Ranges

Topology 1: Star Topology

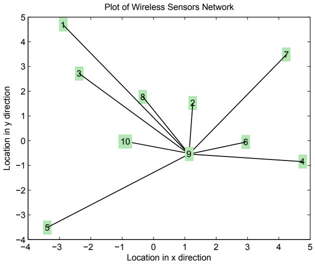

Figure: Network with a Star Topology: Solid lines indicating that there is message communication between nodes. In this network, Node 9 can receive information from all other nodes. Thus Node 9 is the central unit.

- the **phenomenon** is observed by a number of sensors organized as a star
- multiple sensors make measurements
- measurements are transmitted to a fusion center (no messages losses are assumed)

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Topology 2: General Topology

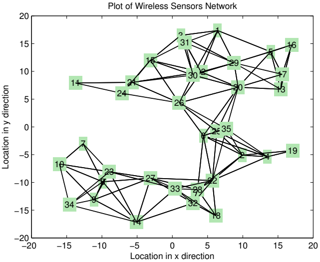

Figure: Network with a Arbitrary Topology: Solid lines indicating that there is message communication between nodes. In this network, there is no node acting as fusion center.

- the **phenomenon** is observed by a number of sensors organized arbitrarily
- multiple sensors make measurements
- measurements are not transmitted to a fusion center
  - indeed, no fusion center. every node is a sort of local fusion center

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Model of the measurements for one sensor

- to start with, we consider only one sensor
- **linear** measurements (i.e., measurements and the parameters are related linearly) with **noise** or **measurement errors**

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{v} \tag{1}$$

- $\mathbf{y}$: sensor measurement(s)
- $\mathbf{H}$: a known matrix
- $\mathbf{x}$: what we want to estimate
- $\mathbf{v}$: unknown noise or measurement error

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- $\mathbf{y}$: sensor measurement(s)
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- $\mathbf{x}$: what we want to estimate
- $\mathbf{v}$: unknown noise or measurement error
- goal: how to estimate **x** out of the measurement **y** ?

Model of the Estimator

linear estimator, i.e.,

$$\hat{\mathbf{x}}(\mathbf{L}) = \mathbf{L}\mathbf{y}$$

- $\mathbf{y}$: sensor measurement(s)
- $\hat{\mathbf{x}}(\mathbf{L})$: estimator of $\mathbf{x}$, dependent on $\mathbf{L}$
- we need to compute a good estimate $\hat{\mathbf{x}}(\cdot) \Rightarrow$ what matrix $\mathbf{L}$ to be used ?
- **performance criterion** for computing $\mathbf{L}$?

Mean Squared Error (MSE) to Chose L

a good estimate $\hat{\mathbf{x}}(\cdot)$ is found by considering the **MSE**, which is given by the trace of **error covariance matrix** $\mathbf{C}$ of the estimator

- in particular, for fixed $\mathbf{L}$, MSE is defined as

$$\begin{aligned} \mathrm{MSE}(\mathbf{L}) &= \mathsf{Tr}\left\{\mathbf{C}(\mathbf{L})\right\} \\ &= \mathsf{Tr}\left\{\mathsf{E}\left\{(\hat{\mathbf{x}}(\mathbf{L}) - \mathbf{x})(\hat{\mathbf{x}}(\mathbf{L}) - \mathbf{x})^{\mathsf{T}}\right\}\right\} \\ &= \textstyle\sum_{i=1}^{N} \mathsf{E}(\hat{x}_i(\mathbf{L}) - x_i)^2 \end{aligned}$$

- let $\mathbf{L}^{\star} = \arg\min_{\mathbf{L}} \mathrm{MSE}(\mathbf{L})$

- then, $\hat{\mathbf{x}} = \mathbf{L}^{\star}\mathbf{y}$ is called the **linear minimum MSE (LMMSE)** estimate of $\mathbf{x}$

LMMSE Estimate

Proposition 1: Consider a random variable $\mathbf{x}$ being observed by a sensor that generate measurements of the form (1), i.e., $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{v}$. Then LMMSE estimator of $\mathbf{x}$ given $\mathbf{y}$ is given by

$$\hat{\mathbf{x}} = \underbrace{\mathbf{P}\mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}}_{\mathbf{L}^{\star}}\mathbf{y} \ , \qquad (2)$$

where

$$\mathbf{P} = \left(\mathbf{R}_{\mathbf{x}}^{-1} + \mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{H}\right)^{-1} \ ,$$

$\mathbf{R}_{\mathbf{x}}$ is the covariance matrix of $\mathbf{x}$, and $\mathbf{R}_{\mathbf{v}}$ is the noise covariance matrix.

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- we need to show that $\mathbf{L}^{\star} = \mathbf{P}\mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}$

LMMSE Estimate Proof

advanced topic, no requested to the exam

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Proof:

preliminaries:

$$
\begin{aligned}
&(1) \quad \mathbf{A} + \mathbf{B} \succeq \mathbf{B} \text{ when } \mathbf{A} \succeq \mathbf{0} \\
&(2) \quad \mathbf{A} \succeq \mathbf{B} \Rightarrow \mathsf{Tr}(\mathbf{L}) \geq \mathsf{Tr}(\mathbf{B}) \\
&(3) \quad (\mathbf{A} + \mathbf{B}\mathbf{C})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{I} + \mathbf{C}\mathbf{A}^{-1}\mathbf{B}\right)^{-1}\mathbf{C}\mathbf{A}^{-1}
\end{aligned}
$$

LMMSE Estimate Proof

Proof:

$$
\begin{aligned}
\mathbf{C}(\mathbf{L}) &= \mathsf{E}\left\{ (\hat{\mathbf{x}}(\mathbf{L}) - \mathbf{x})(\hat{\mathbf{x}}(\mathbf{L}) - \mathbf{x})^{\mathsf{T}} \right\} = \mathsf{E}\left\{ (\mathbf{L}\mathbf{y} - \mathbf{x})(\mathbf{L}\mathbf{y} - \mathbf{x})^{\mathsf{T}} \right\} \\
&= \mathsf{E}\left\{ (\mathbf{L}\mathbf{H} - \mathbf{I})\mathbf{x}\mathbf{x}^{\mathsf{T}}(\mathbf{L}\mathbf{H} - \mathbf{I})^{\mathsf{T}} + \mathbf{L}\mathbf{v}\mathbf{v}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}} \right\} = (\mathbf{L}\mathbf{H} - \mathbf{I})\mathbf{R}_{\mathbf{x}}(\mathbf{L}\mathbf{H} - \mathbf{I})^{\mathsf{T}} + \mathbf{L}\mathbf{R}_{\mathbf{v}}\mathbf{L}^{\mathsf{T}} \\
&= \mathbf{L}\left(\mathbf{H}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}} + \mathbf{R}_{\mathbf{v}}\right)\mathbf{L}^{\mathsf{T}} - \mathbf{L}\mathbf{H}\mathbf{R}_{\mathbf{x}} - \mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}} + \mathbf{R}_{\mathbf{x}} \\
&= \left(\mathbf{L} - \mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\left(\mathbf{H}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}} + \mathbf{R}_{\mathbf{v}}\right)^{-1}\right)\left(\mathbf{H}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}} + \mathbf{R}_{\mathbf{v}}\right)\left(\mathbf{L} - \mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\left(\mathbf{H}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}} + \mathbf{R}_{\mathbf{v}}\right)^{-1}\right)^{\mathsf{T}} \\
&\qquad + \mathbf{R}_{\mathbf{x}} - \mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\left(\mathbf{H}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}} + \mathbf{R}_{\mathbf{v}}\right)^{-1}\mathbf{H}\mathbf{R}_{\mathbf{x}} \\
&\succeq \mathbf{R}_{\mathbf{x}} - \mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\left(\mathbf{H}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}} + \mathbf{R}_{\mathbf{v}}\right)^{-1}\mathbf{H}\mathbf{R}_{\mathbf{x}} \qquad (3)
\end{aligned}
$$

LMMSE Estimate Proof

Proof:

$$\begin{aligned}\mathbf{C}(\mathbf{L}) &= \mathsf{E}\left\{(\hat{\mathbf{x}}(\mathbf{L})-\mathbf{x})(\hat{\mathbf{x}}(\mathbf{L})-\mathbf{x})^{\mathsf{T}}\right\} = \mathsf{E}\left\{(\mathbf{L}\mathbf{y}-\mathbf{x})(\mathbf{L}\mathbf{y}-\mathbf{x})^{\mathsf{T}}\right\} \\
&= \mathsf{E}\left\{(\mathbf{L}\mathbf{H}-\mathbf{I})\mathbf{x}\mathbf{x}^{\mathsf{T}}(\mathbf{L}\mathbf{H}-\mathbf{I})^{\mathsf{T}} + \mathbf{L}\mathbf{v}\mathbf{v}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}}\right\} = (\mathbf{L}\mathbf{H}-\mathbf{I})\mathbf{R}_{\mathbf{x}}(\mathbf{L}\mathbf{H}-\mathbf{I})^{\mathsf{T}} + \mathbf{L}\mathbf{R}_{\mathbf{v}}\mathbf{L}^{\mathsf{T}} \\
&= \mathbf{L}\left(\mathbf{H}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}} + \mathbf{R}_{\mathbf{v}}\right)\mathbf{L}^{\mathsf{T}} - \mathbf{L}\mathbf{H}\mathbf{R}_{\mathbf{x}} - \mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\mathbf{L}^{\mathsf{T}} + \mathbf{R}_{\mathbf{x}} \\
&= \left(\mathbf{L}-\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\left(\mathbf{H}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}+\mathbf{R}_{\mathbf{v}}\right)^{-1}\right)\left(\mathbf{H}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}+\mathbf{R}_{\mathbf{v}}\right)\left(\mathbf{L}-\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\left(\mathbf{H}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}+\mathbf{R}_{\mathbf{v}}\right)^{-1}\right)^{\mathsf{T}} \\
&\qquad + \mathbf{R}_{\mathbf{x}} - \mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\left(\mathbf{H}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}+\mathbf{R}_{\mathbf{v}}\right)^{-1}\mathbf{H}\mathbf{R}_{\mathbf{x}} \\
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the lower bound is achieved when

$$\begin{aligned}\mathbf{L} &= \mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\left(\mathbf{H}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}+\mathbf{R}_{\mathbf{v}}\right)^{-1} \\
&= \mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\left(\mathbf{R}_{\mathbf{v}}^{-1} - \mathbf{R}_{\mathbf{v}}^{-1}\mathbf{H}\left(\mathbf{I}+\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{H}\right)^{-1}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\right) \\
&= \left(\mathbf{I} - \mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{H}\left(\mathbf{I}+\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{H}\right)^{-1}\right)\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1} \\
&= \left(\mathbf{I}+\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{H}\right)^{-1}\mathbf{R}_{\mathbf{x}}\mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1} = \left(\mathbf{R}_{\mathbf{x}}^{-1}+\mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{H}\right)^{-1}\mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1} = \mathbf{P}\mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1} \ \square\end{aligned}$$

LMMSE Estimate

recap:

Consider the linear system of measurements given in (1), i.e., $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{v}$. Let $\hat{\mathbf{x}}$ denote the LMMSE estimator of $\mathbf{x}$ given $\mathbf{y}$. Then we have

$$\mathbf{P}^{-1}\hat{\mathbf{x}} = \mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{y} \ , \tag{4}$$

where

$$\mathbf{P} = \left(\mathbf{R}_{\mathbf{x}}^{-1} + \mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{H}\right)^{-1} = \text{error covariance of } \hat{\mathbf{x}}.$$

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- relation (4) has been derived for **the case of one sensor**
- in **the case of multiple sensors**, relation (4) suggests the possibility of **combining local estimates directly**

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- in **the case of multiple sensors**, relation (4) suggests the possibility of **combining local estimates directly**
- **no** need to sending all the measurements to a **central data processing**

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- relation (4) has been derived for **the case of one sensor**
- in **the case of multiple sensors**, relation (4) suggests the possibility of **combining local estimates directly**
- **no** need to sending all the measurements to a **central data processing**
- this is called **static sensor fusion**

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Static Sensor Fusion, Star Topology

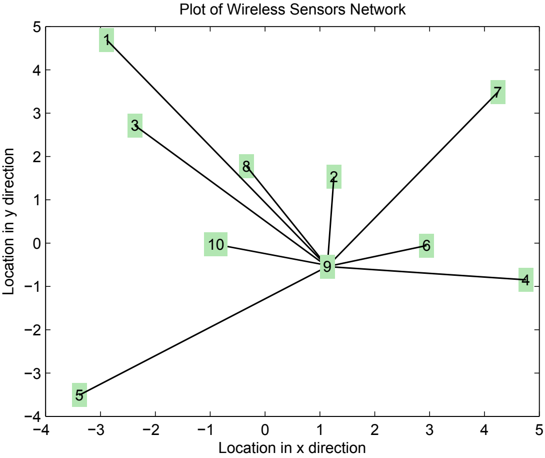

Figure: Network with a Star Topology: Solid lines indicating that there is message communication between nodes. In this network, Node 9 can receive information from all other nodes. Thus Node 9 is the central unit.

- now we move to a case of **many sensors in a star topology**

Static Sensor Fusion, Star Topology

Proposition 2: Consider a random variable $\mathbf{x}$ being observed by K sensors that generate measurements of the form

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x} + \mathbf{v}_k, \quad k = 1, \ldots, K \ . \tag{5}$$

- $\mathbf{y}_k$: kth sensor measurement(s)
- $\mathbf{H}_k$: a matrix known to the kth sensor
- $\mathbf{x}$: what we want to estimate
- $\mathbf{v}_k$: noise or measurement error at kth sensor, $\mathbf{v}_k$ and $\mathbf{v}_j$ $(j \neq k)$ are uncorelated

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Let $\hat{\mathbf{x}}$ denote the LMMSE estimator of $\mathbf{x}$ given $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_K)$, as obtained at the fusion center. Then

$$\mathbf{P}^{-1}\hat{\mathbf{x}} = \textstyle\sum_{k=1}^{K} \mathbf{P}_k^{-1} \hat{\mathbf{x}}_k \ , \ \text{where} \tag{6}$$

where $\mathbf{P}$ is the estimate error covariance corresponding to $\hat{\mathbf{x}}$ and $\mathbf{P}_k$ is the error covariance corresponding to $\hat{\mathbf{x}}_k$.

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where $\mathbf{P}$ is the estimate error covariance corresponding to $\hat{\mathbf{x}}$ and $\mathbf{P}_k$ is the error covariance corresponding to $\hat{\mathbf{x}}_k$. Furthermore,

$$\mathbf{P}^{-1} = -(K-1)\mathbf{R}_{\mathbf{x}}^{-1} + \textstyle\sum_{k=1}^{K} \mathbf{P}_k^{-1} \ , \tag{7}$$

$\mathbf{R}_{\mathbf{x}}$ is the covariance matrix of $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_K)$.

Proof of Proposition 2

Proof: Note that overall linear system is given by

$$\underbrace{\begin{bmatrix}\mathbf{y}_1\\ \vdots \\ \mathbf{y}_K\end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix}\mathbf{H}_1\\ \vdots \\ \mathbf{H}_K\end{bmatrix}}_{\mathbf{H}} \mathbf{x} + \underbrace{\begin{bmatrix}\mathbf{v}_1\\ \vdots \\ \mathbf{v}_K\end{bmatrix}}_{\mathbf{v}} \tag{8}$$

Now use Proposition 1

$$\mathbf{P}^{-1}\hat{\mathbf{x}} = \mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{y} = \begin{bmatrix}\mathbf{H}_1^{\mathsf{T}} \cdots \mathbf{H}_K^{\mathsf{T}}\end{bmatrix} \begin{bmatrix}\mathbf{R}_{\mathbf{v}_1}^{-1} & 0 & \cdots & 0\\ 0 & \mathbf{R}_{\mathbf{v}_2}^{-1} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{R}_{\mathbf{v}_K}^{-1}\end{bmatrix} \begin{bmatrix}\mathbf{y}_1\\ \vdots \\ \mathbf{y}_K\end{bmatrix} \tag{9}$$

$$= \textstyle\sum_{k=1}^{K} \mathbf{H}_k^{\mathsf{T}}\mathbf{R}_{\mathbf{v}_k}^{-1}\mathbf{y}_k \tag{10}$$

$$= \textstyle\sum_{k=1}^{K} \mathbf{P}_k^{-1}\hat{\mathbf{x}}_k \tag{11}$$

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Moreover, from Proposition 1

$$\mathbf{P}^{-1} = \mathbf{R}_{\mathbf{x}}^{-1} + \underbrace{\mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{v}}^{-1}\mathbf{H}} \tag{12}$$

$$= \mathbf{R}_{\mathbf{x}}^{-1} + \textstyle\sum_{k=1}^{K} \underbrace{\mathbf{H}_k^{\mathsf{T}}\mathbf{R}_{\mathbf{v}_k}^{-1}\mathbf{H}_k} \tag{13}$$

$$= \mathbf{R}_{\mathbf{x}}^{-1} + \textstyle\sum_{k=1}^{K} \left(\mathbf{P}_k^{-1} - \mathbf{R}_{\mathbf{x}}^{-1}\right) = -(K-1)\mathbf{R}_{\mathbf{x}}^{-1} + \sum_{k=1}^{K} \mathbf{P}_k^{-1}\ , \tag{14}$$

Static Sensor Fusion from Multiple Sensors

- by Proposition 2, **complexity** of the **fusion center goes down** considerably

- **some computational load is delegated** to the distributed **sensors**

- **each estimate is weighted** by the inverse of the error covariance matrix

- the **higher the confidence** we have in a particular sensor, the **higher the trust** we place in it.

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Sequential Measurements from One Sensor

Proposition 3: Consider a phenomenon $\mathbf{x}$ evolving in time (indexed by n) according to

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Every time step sensor generates a measurement of the form

$$\mathbf{y}_n = \mathbf{C}\mathbf{x}_n + \mathbf{v}_n$$

- $\mathbf{w}_n$: white zero mean Gaussian with covariance matrix $\mathsf{E}\{\mathbf{w}_n\mathbf{w}_n^{\mathsf{T}}\} = \mathbf{Q}$
- $\mathbf{v}_n$: white zero mean Gaussian with covariance matrix $\mathsf{E}\{\mathbf{v}_n\mathbf{v}_n^{\mathsf{T}}\} = \mathbf{R}$
- $\mathbf{A}$: a known nonsingular matrix
- $\mathbf{C}$: a known matrix

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Then we have

$$\hat{\mathbf{x}}_{n|n-1} = \mathbf{A}\hat{\mathbf{x}}_{n-1|n-1} \tag{15}$$

$$\mathbf{P}_{n|n-1} = \mathbf{A}\mathbf{P}_{n-1|n-1}\mathbf{A}^{\mathsf{T}} + \mathbf{Q} \tag{16}$$

- $\hat{\mathbf{x}}_{n-1|n-1}$: estimate of $\mathbf{x}_{n-1}$ given $\mathbf{z} = (\mathbf{y}_0, \ldots, \mathbf{y}_{n-1})$
- $\mathbf{P}_{n-1|n-1}$: corresponding error covariance matrix

Proof of Proposition 3

$\hat{\mathbf{x}}_{n-1|n-1}$: estimate of $\mathbf{x}_{n-1}$ given $\mathbf{z} = (\mathbf{y}_0, \ldots, \mathbf{y}_{n-1})$?

$$\mathbf{y}_{n-1} = \mathbf{C}\mathbf{x}_{n-1} + \mathbf{v}_{n-1} \tag{17}$$

$$\mathbf{y}_{n-2} = \mathbf{C}\mathbf{x}_{n-2} + \mathbf{v}_{n-2} = \mathbf{C}(\mathbf{A}^{-1}(\mathbf{x}_{n-1} - \mathbf{w}_{n-2})) + \mathbf{v}_{n-2} \tag{18}$$

$$= \mathbf{C}\mathbf{A}^{-1}\mathbf{x}_{n-1} + \left(\mathbf{v}_{n-2} - \mathbf{C}\mathbf{A}^{-1}\mathbf{w}_{n-2}\right) \tag{19}$$

$$\vdots$$

$$\mathbf{y}_0 = \mathbf{C}\mathbf{A}^{-(n-1)}\mathbf{x}_{n-1} + \left(\mathbf{v}_0 - \mathbf{C}\mathbf{A}^{-(n-1)}\mathbf{w}_{n-2} - \cdots - \mathbf{C}\mathbf{A}^{-1}\mathbf{w}_0\right) \tag{20}$$

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i.e., the overall linear system is given by

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From Proposition 1

$$\mathbf{P}_{n-1|n-1}^{-1}\hat{\mathbf{x}}_{n-1|n-1} = \mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{z} \text{ , where}$$

$$\mathbf{P}_{n-1|n-1} = \left(\mathbf{R}_{\mathbf{x}_{n-1}}^{-1} + \mathbf{H}^{\mathsf{T}}\mathbf{R}_{\mathbf{u}}^{-1}\mathbf{H}\right)^{-1}$$

Proof of Proposition 3

Question: how to show that $\hat{\mathbf{x}}_{n|n-1} = \mathbf{A}\hat{\mathbf{x}}_{n-1|n-1}$?

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Question: how to show that $\hat{\mathbf{x}}_{n|n-1} = \mathbf{A}\hat{\mathbf{x}}_{n-1|n-1}$?

Answer:
use a well known result: MMSE estimate $\hat{\mathbf{x}}$ of a random variable $\mathbf{x}$ given a random variable $\mathbf{y}$ is $\mathsf{E}\{\mathbf{x}|\mathbf{y}\}$.

$$\begin{aligned}\hat{\mathbf{x}}_{n|n-1} &= \mathsf{E}\left\{\mathbf{x}_n|(\mathbf{y}_0,\ldots,\mathbf{y}_{n-1})\right\} \\ &= \mathsf{E}\left\{\mathbf{A}\mathbf{x}_{n-1} + \mathbf{w}_{n-1}|\mathbf{z})\right\} \\ &= \mathbf{A}\mathsf{E}\left\{\mathbf{x}_{n-1}|\mathbf{z}\right\} \\ &= \mathbf{A}\hat{\mathbf{x}}_{n-1|n-1}\end{aligned}$$

Proof of Proposition 3

Question: how to show that $\mathbf{P}_{n|n-1} = \mathbf{A}\mathbf{P}_{n-1|n-1}\mathbf{A}^{\mathsf{T}} + \mathbf{Q}$?

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Answer:
by the definition of error covariance, we already have:

$$\mathbf{P}_{n-1|n-1} = \mathsf{E}\{(\hat{\mathbf{x}}_{n-1|n-1} - \mathbf{x}_{n-1})(\hat{\mathbf{x}}_{n-1|n-1} - \mathbf{x}_{n-1})^{\mathsf{T}}\} \qquad (22)$$

$$\begin{aligned}
\mathbf{P}_{n|n-1} &= \mathsf{E}\{(\hat{\mathbf{x}}_{n|n-1} - \mathbf{x}_{n})(\hat{\mathbf{x}}_{n|n-1} - \mathbf{x}_{n})^{\mathsf{T}}\} \\
&= \mathsf{E}\{(\mathbf{A}\hat{\mathbf{x}}_{n-1|n-1} - \mathbf{A}\mathbf{x}_{n-1} - \mathbf{w}_{n-1})(\mathbf{A}\hat{\mathbf{x}}_{n-1|n-1} - \mathbf{A}\mathbf{x}_{n-1} - \mathbf{w}_{n-1})^{\mathsf{T}}\} \\
&= \mathsf{E}\{\mathbf{A}(\hat{\mathbf{x}}_{n-1|n-1} - \mathbf{x}_{n-1})(\hat{\mathbf{x}}_{n-1|n-1} - \mathbf{x}_{n-1})^{\mathsf{T}}\mathbf{A}^{\mathsf{T}} + \mathbf{w}_{n-1}\mathbf{w}_{n-1}^{\mathsf{T}}\} \\
&= \mathbf{A}\mathbf{P}_{n-1|n-1}\mathbf{A}^{\mathsf{T}} + \mathbf{Q}
\end{aligned}$$

Sequential Measurements from One Sensor

Question: $\hat{\mathrm{x}}_{n|n}$, the MMSE estimate of $\mathbf{x}_n$ given $(\mathbf{y}_0, \ldots, \mathbf{y}_{n-1}, \mathbf{y}_n) = (\mathbf{z}, \mathbf{y}_n)$

Sequential Measurements from One Sensor

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Answer: apply Proposition 2 (Static Sensor Fusion) in a straightforward manner

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we **already know** $\hat{\mathbf{x}}_{n|n-1}$, i.e., the estimate of $\mathbf{x}_n$ given $\mathbf{z}$
we **already know** $\mathbf{P}_{n|n-1}$, the corresponding error covariance matrix

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we **need** the estimate of $\mathbf{x}_n$ given $\mathbf{y}_n$, denote it by $\hat{\mathbf{x}}$
we **need** the corresponding error covariance matrix, denote it by $\mathbf{M}$

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because $\mathbf{y}_n = \mathbf{C}\mathbf{x}_n + \mathbf{v}_n$, from Proposition 1, we have

$$\hat{\mathbf{x}} = \mathbf{M}\mathbf{C}^{\mathsf{T}}\mathbf{R}^{-1}\mathbf{y}_n \text{ , where } \mathbf{M} = \left(\mathbf{R}_{\mathbf{x}_n}^{-1} + \mathbf{C}^{\mathsf{T}}\mathbf{R}^{-1}\mathbf{C}\right)^{-1}$$

Sequential Measurements from One Sensor

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from Proposition 2

$$\mathbf{P}_{n|n}^{-1}\hat{\mathbf{x}}_{n|n} = \mathbf{P}_{n|n-1}^{-1}\hat{\mathbf{x}}_{n|n-1} + \mathbf{M}^{-1}\hat{\mathbf{x}} \text{ , where} \quad (23)$$

$$\mathbf{P}_{n|n}^{-1} = -\mathbf{R}_{\mathbf{x}_n}^{-1} + \mathbf{P}_{n|n-1}^{-1} + \mathbf{M}^{-1} \quad (24)$$

Sequential Measurements from One Sensor

- **time** and **measurement** update steps of the **Kalman filter**
- **Kalman filter** can be seen to be a **combination of estimators**
- **optimality** of the **Kalman filter** in the **minimum mean squared** sense

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Dynamic Sensor Fusion

Consider a phenomenon $\mathbf{x}$ evolving in time (indexed by n) according to the law

$$\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n + \mathbf{w}_n$$

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- **multiple sensors** that generate measurements about the random variable that is evolving in time

- **Question:** how to **fuse data** from all the sensors for an estimate of the state $\mathbf{x}_n$ at time step n

Dynamic Sensor Fusion, Centralized Setup

- at every time step n, all the sensors **transmit** their measurements $\mathbf{y}_{n,k}$ to **a central node**
- the central node implements a fusion mechanism
- however, there are two reasons why this **may not be the preferred** implementation

(1) number of sensors increases $\Rightarrow$ computational effort required at the central node increases (bear some of the computational burden at sensors)
(2) the sensors may not be able to transmit at every time step (transmit local processed information rather that raw measurements)

Dynamic Sensor Fusion, Centralized Setup (Transmitting Local Estimates)

- assume that the sensors can transmit at every time step
- reducing the computational burden at the central node ?

Dynamic Sensor Fusion, Centralized Setup (Transmitting Local Estimates)

let $\mathbf{y}_k = (\mathbf{y}_{0,k}, \mathbf{y}_{1,k}, \ldots, \mathbf{y}_{n,k})$ denote the measurements from sensor k that is used to estimate $\mathbf{x}_n$

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potential method 1

the overall linear system is given by

$$\underbrace{\begin{bmatrix} \mathbf{y}_{n,k} \\ \mathbf{y}_{n-1,k} \\ \vdots \\ \mathbf{y}_{0,k} \end{bmatrix}}_{\mathbf{y}_k} = \underbrace{\begin{bmatrix} \mathbf{C}_k \\ \mathbf{C}_k\mathbf{A}^{-1} \\ \vdots \\ \mathbf{C}_k\mathbf{A}^{-n} \end{bmatrix}}_{\mathbf{H}_k} \mathbf{x}_n + \underbrace{\begin{bmatrix} \mathbf{v}_{n,k} \\ \mathbf{v}_{n-1,k} - \mathbf{C}_k\mathbf{A}^{-1}\mathbf{w}_{n-1} \\ \vdots \\ \mathbf{v}_{0,k} - \cdots - \mathbf{C}_k\mathbf{A}^{-1}\mathbf{w}_0 \end{bmatrix}}_{\mathbf{v}_k} \tag{25}$$

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- process noise $\mathbf{w}_n$ appears in the noise $\Rightarrow$ the **measurement noises** $\mathbf{v}_k$ are **not independent** as desired
- $\mathbf{v}_k$ are not independent $\Rightarrow$ the **noise is correlated**
- $\Rightarrow$ **Proposition 2 does not apply** for combining local estimates

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potential method 2: estimate of $\mathbf{x}_0$ known $\Rightarrow$ estimate of $\mathbf{x}_n$ known

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- the **measurement noises** $\mathbf{v}_k$ are **independent** as desired
- $\Rightarrow$ **Proposition 2 does apply** for combining local estimates
- **vectors** transmitted from sensors are **increasing in dimension** as the time step n increases.

Dynamic Sensor Fusion, Centralized Setup (Drawbacks)

- practically, it is not feasible to combine local estimates from method 2 to obtain the global estimate
- i.e., lots of communication overhead
- if there is no process noise, then the method 1 will work
- however, in general it is not possible

Dynamic Sensor Fusion
Distributed Kalman Filtering

- **recall:** Sequential Measurements from One Sensor

Dynamic Sensor Fusion Distributed Kalman Filtering

- **recall:** Sequential Measurements from One Sensor
- random variable evolution: $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n + \mathbf{w}_n$
- measurements: $\mathbf{y}_n = \mathbf{C}\mathbf{x}_n + \mathbf{v}_n$, where $\mathsf{E}\{\mathbf{v}_n \mathbf{v}_n^\mathsf{T}\} = \mathbf{R}$

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- we have

$$\begin{bmatrix}\mathbf{y}_n \\ \mathbf{z}\end{bmatrix} = \begin{bmatrix}\mathbf{C} \\ \mathbf{H}\end{bmatrix}\mathbf{x}_n + \begin{bmatrix}\mathbf{v}_n \\ \mathbf{u}\end{bmatrix} \tag{27}$$

$$\hat{\mathbf{x}} = \mathbf{M}\mathbf{C}^{\mathsf{T}}\mathbf{R}^{-1}\mathbf{y}_n \text{ , where } \mathbf{M} = \left(\mathbf{R}_{\mathbf{x}_n}^{-1} + \mathbf{C}^{\mathsf{T}}\mathbf{R}^{-1}\mathbf{C}\right)^{-1}$$

$$\mathbf{P}_{n|n}^{-1}\hat{\mathbf{x}}_{n|n} = \mathbf{P}_{n|n-1}^{-1}\hat{\mathbf{x}}_{n|n-1} + \mathbf{C}^{\mathsf{T}}\mathbf{R}^{-1}\mathbf{y}_n \tag{28}$$

$$\mathbf{P}_{n|n}^{-1} = \mathbf{P}_{n|n-1}^{-1} + \mathbf{C}^{\mathsf{T}}\mathbf{R}^{-1}\mathbf{C} \tag{29}$$

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$$\begin{bmatrix} \mathbf{y}_n \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{H} \end{bmatrix} \mathbf{x}_n + \begin{bmatrix} \mathbf{v}_n \\ \mathbf{u} \end{bmatrix} \tag{27}$$

$$\hat{\mathbf{x}} = \mathbf{M}\mathbf{C}^{\mathsf{T}}\mathbf{R}^{-1}\mathbf{y}_n \text{ , where } \mathbf{M} = \left(\mathbf{R}_{\mathbf{x}_n}^{-1} + \mathbf{C}^{\mathsf{T}}\mathbf{R}^{-1}\mathbf{C}\right)^{-1}$$

$$\mathbf{P}_{n|n}^{-1}\hat{\mathbf{x}}_{n|n} = \mathbf{P}_{n|n-1}^{-1}\hat{\mathbf{x}}_{n|n-1} + \mathbf{C}^{\mathsf{T}}\mathbf{R}^{-1}\mathbf{y}_n \tag{28}$$

$$\mathbf{P}_{n|n}^{-1} = \mathbf{P}_{n|n-1}^{-1} + \mathbf{C}^{\mathsf{T}}\mathbf{R}^{-1}\mathbf{C} \tag{29}$$

- the requirements from individual sensors are derived by the equations above

Dynamic Sensor Fusion
Distributed Kalman Filtering

Proposition 4: Consider a random variable $\mathbf{x}_n$ evolving in time as $\mathbf{x}_n = \mathbf{A}\mathbf{x}_{n-1} + \mathbf{w}_{n-1}$ being observed by K sensors in every time step n. Suppose they generate measurements of the form $\mathbf{y}_{n,k} = \mathbf{C}_k\mathbf{x}_n + \mathbf{v}_{n,k}$. Then the global error covariance matrix and the estimate are given in terms of the local covariances and estimates by

$$\mathbf{P}_{n|n}^{-1} = \mathbf{P}_{n|n-1}^{-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} - \mathbf{P}_{n,k|n-1}^{-1}\right)$$

$$\mathbf{P}_{n|n}^{-1}\hat{\mathbf{x}}_{n|n} = \mathbf{P}_{n|n-1}^{-1}\hat{\mathbf{x}}_{n|n-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1}\hat{\mathbf{x}}_{n,k|n} - \mathbf{P}_{n,k|n-1}^{-1}\hat{\mathbf{x}}_{n,k|n-1}\right)$$

Dynamic Sensor Fusion
Distributed Kalman Filtering

Proof: Note that overall linear system is given by

$$\begin{bmatrix} \mathbf{y}_{n,1} \\ \vdots \\ \mathbf{y}_{n,K} \\ \mathbf{z}_1 \\ \vdots \\ \mathbf{z}_K \end{bmatrix} = \begin{bmatrix} \mathbf{C}_1 \\ \vdots \\ \mathbf{C}_K \\ \mathbf{H}_1 \\ \vdots \\ \mathbf{H}_K \end{bmatrix} \mathbf{x}_n + \begin{bmatrix} \mathbf{v}_{n,1} \\ \vdots \\ \mathbf{v}_{n,K} \\ \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_K \end{bmatrix} \triangleq \begin{bmatrix} \mathbf{y}_n \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{H} \end{bmatrix} \mathbf{x}_n + \begin{bmatrix} \mathbf{v}_n \\ \mathbf{u} \end{bmatrix}$$

Lets now simplify $\mathbf{C}^\mathsf{T}\mathbf{R}^{-1}\mathbf{y}_n$

$$\mathbf{C}^\mathsf{T}\mathbf{R}^{-1}\mathbf{y}_n = \begin{bmatrix} \mathbf{C}_1^\mathsf{T} \cdots \mathbf{C}_K^\mathsf{T} \end{bmatrix} \begin{bmatrix} \mathbf{R}_1^{-1} & 0 & \cdots & 0 \\ 0 & \mathbf{R}_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{R}_K^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{y}_{n,1} \\ \vdots \\ \mathbf{y}_{n,K} \end{bmatrix}$$

$$= \sum_{k=1}^{K} \mathbf{C}_k^\mathsf{T}\mathbf{R}_k^{-1}\mathbf{y}_{n,k}$$

$$= \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1}\hat{\mathbf{x}}_{n,k|n} - \mathbf{P}_{n,k|n-1}^{-1}\hat{\mathbf{x}}_{n,k|n-1} \right) \quad (30)$$

$$\mathbf{C}^\mathsf{T}\mathbf{R}^{-1}\mathbf{C} = \sum_{k=1}^{K} \mathbf{C}_k^\mathsf{T}\mathbf{R}_k^{-1}\mathbf{C}_k$$

$$= \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} - \mathbf{P}_{n,k|n-1}^{-1} \right) \quad (31)$$

Dynamic Sensor Fusion
Distributed Kalman Filtering

recap:

$$\mathbf{P}_{n|n}^{-1} = \mathbf{P}_{n|n-1}^{-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} - \mathbf{P}_{n,k|n-1}^{-1} \right)$$

$$\mathbf{P}_{n|n}^{-1} \hat{\mathbf{x}}_{n|n} = \mathbf{P}_{n|n-1}^{-1} \hat{\mathbf{x}}_{n|n-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} \hat{\mathbf{x}}_{n,k|n} - \mathbf{P}_{n,k|n-1}^{-1} \hat{\mathbf{x}}_{n,k|n-1} \right)$$

Based on the result above → **two architectures** for dynamic sensor fusion

- **method 1: more** computation at the fusion center, **less** communication overhead
- **method 2: less** computation at the fusion center, **more** communication overhead

Dynamic Sensor Fusion Distributed Kalman Filtering (method 1)

say $n = 0$....what will happen ?

$$\mathbf{P}_{n|n}^{-1} = \mathbf{P}_{n|n-1}^{-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} - \mathbf{P}_{n,k|n-1}^{-1} \right)$$

$$\mathbf{P}_{n|n}^{-1} \hat{\mathbf{x}}_{n|n} = \mathbf{P}_{n|n-1}^{-1} \hat{\mathbf{x}}_{n|n-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} \hat{\mathbf{x}}_{n,k|n} - \mathbf{P}_{n,k|n-1}^{-1} \hat{\mathbf{x}}_{n,k|n-1} \right)$$

Dynamic Sensor Fusion
Distributed Kalman Filtering (method 1)

say $n = 0$....what will happen ?

$$\mathbf{P}_{n|n}^{-1} = \mathbf{P}_{n|n-1}^{-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} - \mathbf{P}_{n,k|n-1}^{-1} \right)$$

$$\mathbf{P}_{n|n}^{-1} \hat{\mathbf{x}}_{n|n} = \mathbf{P}_{n|n-1}^{-1} \hat{\mathbf{x}}_{n|n-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} \hat{\mathbf{x}}_{n,k|n} - \mathbf{P}_{n,k|n-1}^{-1} \hat{\mathbf{x}}_{n,k|n-1} \right)$$

$\mathbf{y}_{0,1}$	$\mathbf{y}_{1,1}$	$\mathbf{y}_{2,1}$	$\mathbf{y}_{3,1}$

sensor 1 measurements

$\mathbf{x}_0$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$

what we want to estimate

$\mathbf{y}_{0,2}$	$\mathbf{y}_{1,2}$	$\mathbf{y}_{2,2}$	$\mathbf{y}_{3,2}$

sensor 2 measurements

Dynamic Sensor Fusion
Distributed Kalman Filtering (method 1)

say $n = 0$....what will happen ?

$$\mathbf{P}_{n|n}^{-1} = \mathbf{P}_{n|n-1}^{-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} - \mathbf{P}_{n,k|n-1}^{-1} \right)$$

$$\mathbf{P}_{n|n}^{-1} \hat{\mathbf{x}}_{n|n} = \mathbf{P}_{n|n-1}^{-1} \hat{\mathbf{x}}_{n|n-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} \hat{\mathbf{x}}_{n,k|n} - \mathbf{P}_{n,k|n-1}^{-1} \hat{\mathbf{x}}_{n,k|n-1} \right)$$

$\mathbf{y}_{0,1}$	$\mathbf{y}_{1,1}$	$\mathbf{y}_{2,1}$	$\mathbf{y}_{3,1}$

sensor 1 measurements

$\mathbf{x}_0$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$

what we want to estimate

$\mathbf{y}_{0,2}$	$\mathbf{y}_{1,2}$	$\mathbf{y}_{2,2}$	$\mathbf{y}_{3,2}$

sensor 2 measurements

Dynamic Sensor Fusion
Distributed Kalman Filtering (method 1)

say $n = 0$....what will happen ?

$$\mathbf{P}_{n|n}^{-1} = \mathbf{P}_{n|n-1}^{-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} - \mathbf{P}_{n,k|n-1}^{-1} \right)$$

$$\mathbf{P}_{n|n}^{-1} \hat{\mathbf{x}}_{n|n} = \mathbf{P}_{n|n-1}^{-1} \hat{\mathbf{x}}_{n|n-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} \hat{\mathbf{x}}_{n,k|n} - \mathbf{P}_{n,k|n-1}^{-1} \hat{\mathbf{x}}_{n,k|n-1} \right)$$

$\mathbf{y}_{0,1}$	$\mathbf{y}_{1,1}$	$\mathbf{y}_{2,1}$	$\mathbf{y}_{3,1}$

sensor 1 measurements

$\mathbf{x}_0$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$

what we want to estimate

$\mathbf{y}_{0,2}$	$\mathbf{y}_{1,2}$	$\mathbf{y}_{2,2}$	$\mathbf{y}_{3,2}$

sensor 2 measurements

sensor 1 / sensor 2

fusion center

Dynamic Sensor Fusion
Distributed Kalman Filtering (method 1)

say $n = 0$....what will happen ?

$$\mathbf{P}_{n|n}^{-1} = \mathbf{P}_{n|n-1}^{-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} - \mathbf{P}_{n,k|n-1}^{-1} \right)$$

$$\mathbf{P}_{n|n}^{-1} \hat{\mathbf{x}}_{n|n} = \mathbf{P}_{n|n-1}^{-1} \hat{\mathbf{x}}_{n|n-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} \hat{\mathbf{x}}_{n,k|n} - \mathbf{P}_{n,k|n-1}^{-1} \hat{\mathbf{x}}_{n,k|n-1} \right)$$

$\mathbf{y}_{0,1}$	$\mathbf{y}_{1,1}$	$\mathbf{y}_{2,1}$	$\mathbf{y}_{3,1}$

sensor 1 measurements

$\mathbf{x}_0$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$

what we want to estimate

$\mathbf{y}_{0,2}$	$\mathbf{y}_{1,2}$	$\mathbf{y}_{2,2}$	$\mathbf{y}_{3,2}$

sensor 2 measurements

sensor 1 / sensor 2

$\mathbf{P}_{0,1|0}^{-1}, \hat{\mathbf{x}}_{0,1|0}$
$\mathbf{P}_{0,2|0}^{-1}, \hat{\mathbf{x}}_{0,2|0}$

fusion center

Dynamic Sensor Fusion
Distributed Kalman Filtering (method 1)

say $n = 0$....what will happen ?

$$\mathbf{P}_{n|n}^{-1} = \mathbf{P}_{n|n-1}^{-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} - \mathbf{P}_{n,k|n-1}^{-1} \right)$$

$$\mathbf{P}_{n|n}^{-1}\hat{\mathbf{x}}_{n|n} = \mathbf{P}_{n|n-1}^{-1}\hat{\mathbf{x}}_{n|n-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1}\hat{\mathbf{x}}_{n,k|n} - \mathbf{P}_{n,k|n-1}^{-1}\hat{\mathbf{x}}_{n,k|n-1} \right)$$

$\mathbf{y}_{0,1}$	$\mathbf{y}_{1,1}$	$\mathbf{y}_{2,1}$	$\mathbf{y}_{3,1}$

sensor 1 measurements

$\mathbf{x}_0$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$

what we want to estimate

$\mathbf{y}_{0,2}$	$\mathbf{y}_{1,2}$	$\mathbf{y}_{2,2}$	$\mathbf{y}_{3,2}$

sensor 2 measurements

sensor 1 / sensor 2 fusion center

$\mathbf{P}_{0,1|0}^{-1}, \hat{\mathbf{x}}_{0,1|0}$
$\mathbf{P}_{0,2|0}^{-1}, \hat{\mathbf{x}}_{0,2|0}$ $\longmapsto$

Dynamic Sensor Fusion
Distributed Kalman Filtering (method 1)

say $n = 0$....what will happen ?

$$\mathbf{P}_{n|n}^{-1} = \mathbf{P}_{n|n-1}^{-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} - \mathbf{P}_{n,k|n-1}^{-1} \right)$$

$$\mathbf{P}_{n|n}^{-1} \hat{\mathbf{x}}_{n|n} = \mathbf{P}_{n|n-1}^{-1} \hat{\mathbf{x}}_{n|n-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} \hat{\mathbf{x}}_{n,k|n} - \mathbf{P}_{n,k|n-1}^{-1} \hat{\mathbf{x}}_{n,k|n-1} \right)$$

$\mathbf{y}_{0,1}$	$\mathbf{y}_{1,1}$	$\mathbf{y}_{2,1}$	$\mathbf{y}_{3,1}$

sensor 1 measurements

$\mathbf{x}_0$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$

what we want to estimate

$\mathbf{y}_{0,2}$	$\mathbf{y}_{1,2}$	$\mathbf{y}_{2,2}$	$\mathbf{y}_{3,2}$

sensor 2 measurements

sensor 1 / sensor 2

$\mathbf{P}_{0,1|0}^{-1}, \hat{\mathbf{x}}_{0,1|0}$

$\mathbf{P}_{0,2|0}^{-1}, \hat{\mathbf{x}}_{0,2|0}$

$\longmapsto$

fusion center

$$\mathbf{P}_{1,1|0} = \mathbf{A}\mathbf{P}_{0,1|0}\mathbf{A}^{\mathsf{T}} + \mathbf{Q}$$

Dynamic Sensor Fusion
Distributed Kalman Filtering (method 1)

say $n = 0$....what will happen ?

$$\mathbf{P}_{n|n}^{-1} = \mathbf{P}_{n|n-1}^{-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} - \mathbf{P}_{n,k|n-1}^{-1} \right)$$

$$\mathbf{P}_{n|n}^{-1} \hat{\mathbf{x}}_{n|n} = \mathbf{P}_{n|n-1}^{-1} \hat{\mathbf{x}}_{n|n-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} \hat{\mathbf{x}}_{n,k|n} - \mathbf{P}_{n,k|n-1}^{-1} \hat{\mathbf{x}}_{n,k|n-1} \right)$$

$\mathbf{y}_{0,1}$	$\mathbf{y}_{1,1}$	$\mathbf{y}_{2,1}$	$\mathbf{y}_{3,1}$

sensor 1 measurements

$\mathbf{x}_0$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$

what we want to estimate

$\mathbf{y}_{0,2}$	$\mathbf{y}_{1,2}$	$\mathbf{y}_{2,2}$	$\mathbf{y}_{3,2}$

sensor 2 measurements

sensor 1 / sensor 2

$\mathbf{P}_{0,1|0}^{-1}, \hat{\mathbf{x}}_{0,1|0}$

$\mathbf{P}_{0,2|0}^{-1}, \hat{\mathbf{x}}_{0,2|0}$

$\longmapsto$

fusion center

$$\mathbf{P}_{1,1|0} = \mathbf{A}\mathbf{P}_{0,1|0}\mathbf{A}^{\mathsf{T}} + \mathbf{Q}$$

$$\hat{\mathbf{x}}_{1,1|0} = \mathbf{A}\hat{\mathbf{x}}_{0,1|0}$$

Dynamic Sensor Fusion
Distributed Kalman Filtering (method 1)

say $n = 0$....what will happen ?

$$\mathbf{P}_{n|n}^{-1} = \mathbf{P}_{n|n-1}^{-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} - \mathbf{P}_{n,k|n-1}^{-1} \right)$$

$$\mathbf{P}_{n|n}^{-1}\hat{\mathbf{x}}_{n|n} = \mathbf{P}_{n|n-1}^{-1}\hat{\mathbf{x}}_{n|n-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1}\hat{\mathbf{x}}_{n,k|n} - \mathbf{P}_{n,k|n-1}^{-1}\hat{\mathbf{x}}_{n,k|n-1} \right)$$

$\mathbf{y}_{0,1}$	$\mathbf{y}_{1,1}$	$\mathbf{y}_{2,1}$	$\mathbf{y}_{3,1}$

sensor 1 measurements

$\mathbf{x}_0$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$

what we want to estimate

$\mathbf{y}_{0,2}$	$\mathbf{y}_{1,2}$	$\mathbf{y}_{2,2}$	$\mathbf{y}_{3,2}$

sensor 2 measurements

sensor 1 / sensor 2

$\mathbf{P}_{0,1|0}^{-1}, \hat{\mathbf{x}}_{0,1|0}$

$\mathbf{P}_{0,2|0}^{-1}, \hat{\mathbf{x}}_{0,2|0}$

$\longmapsto$

fusion center

$$\mathbf{P}_{1,1|0} = \mathbf{A}\mathbf{P}_{0,1|0}\mathbf{A}^{\mathsf{T}} + \mathbf{Q}$$

$$\hat{\mathbf{x}}_{1,1|0} = \mathbf{A}\hat{\mathbf{x}}_{0,1|0}$$

$$\mathbf{P}_{1,2|0} = \mathbf{A}\mathbf{P}_{0,2|0}\mathbf{A}^{\mathsf{T}} + \mathbf{Q}$$

Dynamic Sensor Fusion
Distributed Kalman Filtering (method 1)

say $n = 0$....what will happen ?

$$\mathbf{P}_{n|n}^{-1} = \mathbf{P}_{n|n-1}^{-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} - \mathbf{P}_{n,k|n-1}^{-1} \right)$$

$$\mathbf{P}_{n|n}^{-1}\hat{\mathbf{x}}_{n|n} = \mathbf{P}_{n|n-1}^{-1}\hat{\mathbf{x}}_{n|n-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1}\hat{\mathbf{x}}_{n,k|n} - \mathbf{P}_{n,k|n-1}^{-1}\hat{\mathbf{x}}_{n,k|n-1} \right)$$

$\mathbf{y}_{0,1}$	$\mathbf{y}_{1,1}$	$\mathbf{y}_{2,1}$	$\mathbf{y}_{3,1}$

sensor 1 measurements

$\mathbf{x}_0$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$

what we want to estimate

$\mathbf{y}_{0,2}$	$\mathbf{y}_{1,2}$	$\mathbf{y}_{2,2}$	$\mathbf{y}_{3,2}$

sensor 2 measurements

sensor 1 / sensor 2

$\mathbf{P}_{0,1|0}^{-1}, \hat{\mathbf{x}}_{0,1|0}$
$\mathbf{P}_{0,2|0}^{-1}, \hat{\mathbf{x}}_{0,2|0}$

$\longmapsto$

fusion center

$$\mathbf{P}_{1,1|0} = \mathbf{A}\mathbf{P}_{0,1|0}\mathbf{A}^{\mathsf{T}} + \mathbf{Q}$$
$$\hat{\mathbf{x}}_{1,1|0} = \mathbf{A}\hat{\mathbf{x}}_{0,1|0}$$
$$\mathbf{P}_{1,2|0} = \mathbf{A}\mathbf{P}_{0,2|0}\mathbf{A}^{\mathsf{T}} + \mathbf{Q}$$
$$\hat{\mathbf{x}}_{1,2|0} = \mathbf{A}\hat{\mathbf{x}}_{0,2|0}$$

Dynamic Sensor Fusion
Distributed Kalman Filtering (method 1)

say $n = 0$....what will happen ?

$$\mathbf{P}_{n|n}^{-1} = \mathbf{P}_{n|n-1}^{-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} - \mathbf{P}_{n,k|n-1}^{-1} \right)$$

$$\mathbf{P}_{n|n}^{-1} \hat{\mathbf{x}}_{n|n} = \mathbf{P}_{n|n-1}^{-1} \hat{\mathbf{x}}_{n|n-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} \hat{\mathbf{x}}_{n,k|n} - \mathbf{P}_{n,k|n-1}^{-1} \hat{\mathbf{x}}_{n,k|n-1} \right)$$

$\mathbf{y}_{0,1}$	$\mathbf{y}_{1,1}$	$\mathbf{y}_{2,1}$	$\mathbf{y}_{3,1}$

sensor 1 measurements

$\mathbf{x}_0$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$

what we want to estimate

$\mathbf{y}_{0,2}$	$\mathbf{y}_{1,2}$	$\mathbf{y}_{2,2}$	$\mathbf{y}_{3,2}$

sensor 2 measurements

sensor 1 / sensor 2

$\mathbf{P}_{0,1|0}^{-1}, \hat{\mathbf{x}}_{0,1|0}$

$\mathbf{P}_{0,2|0}^{-1}, \hat{\mathbf{x}}_{0,2|0}$

$\longmapsto$

fusion center

$$\mathbf{P}_{1,1|0} = \mathbf{A}\mathbf{P}_{0,1|0}\mathbf{A}^{\mathsf{T}} + \mathbf{Q}$$
$$\hat{\mathbf{x}}_{1,1|0} = \mathbf{A}\hat{\mathbf{x}}_{0,1|0}$$
$$\mathbf{P}_{1,2|0} = \mathbf{A}\mathbf{P}_{0,2|0}\mathbf{A}^{\mathsf{T}} + \mathbf{Q}$$
$$\hat{\mathbf{x}}_{1,2|0} = \mathbf{A}\hat{\mathbf{x}}_{0,2|0}$$
$$\mathbf{P}_{1|0} = \mathbf{A}\mathbf{P}_{0|0}\mathbf{A}^{\mathsf{T}} + \mathbf{Q}$$

Dynamic Sensor Fusion
Distributed Kalman Filtering (method 1)

say $n = 0$....what will happen ?

$$\mathbf{P}^{-1}_{n|n} = \mathbf{P}^{-1}_{n|n-1} + \sum_{k=1}^{K} \left(\mathbf{P}^{-1}_{n,k|n} - \mathbf{P}^{-1}_{n,k|n-1} \right)$$

$$\mathbf{P}^{-1}_{n|n} \hat{\mathbf{x}}_{n|n} = \mathbf{P}^{-1}_{n|n-1} \hat{\mathbf{x}}_{n|n-1} + \sum_{k=1}^{K} \left(\mathbf{P}^{-1}_{n,k|n} \hat{\mathbf{x}}_{n,k|n} - \mathbf{P}^{-1}_{n,k|n-1} \hat{\mathbf{x}}_{n,k|n-1} \right)$$

$\mathbf{y}_{0,1}$	$\mathbf{y}_{1,1}$	$\mathbf{y}_{2,1}$	$\mathbf{y}_{3,1}$

sensor 1 measurements

$\mathbf{x}_0$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$

what we want to estimate

$\mathbf{y}_{0,2}$	$\mathbf{y}_{1,2}$	$\mathbf{y}_{2,2}$	$\mathbf{y}_{3,2}$

sensor 2 measurements

sensor 1 / sensor 2

$\mathbf{P}^{-1}_{0,1|0}, \hat{\mathbf{x}}_{0,1|0}$

$\mathbf{P}^{-1}_{0,2|0}, \hat{\mathbf{x}}_{0,2|0}$

$\longmapsto$

fusion center

$$\mathbf{P}_{1,1|0} = \mathbf{A}\mathbf{P}_{0,1|0}\mathbf{A}^{\mathsf{T}} + \mathbf{Q}$$
$$\hat{\mathbf{x}}_{1,1|0} = \mathbf{A}\hat{\mathbf{x}}_{0,1|0}$$
$$\mathbf{P}_{1,2|0} = \mathbf{A}\mathbf{P}_{0,2|0}\mathbf{A}^{\mathsf{T}} + \mathbf{Q}$$
$$\hat{\mathbf{x}}_{1,2|0} = \mathbf{A}\hat{\mathbf{x}}_{0,2|0}$$
$$\mathbf{P}_{1|0} = \mathbf{A}\mathbf{P}_{0|0}\mathbf{A}^{\mathsf{T}} + \mathbf{Q}$$
$$\hat{\mathbf{x}}_{1|0} = \mathbf{A}\hat{\mathbf{x}}_{0|0}$$

Dynamic Sensor Fusion
Distributed Kalman Filtering (method 1)

say $n = 1$....what will happen ?

$$\mathbf{P}_{n|n}^{-1} = \mathbf{P}_{n|n-1}^{-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} - \mathbf{P}_{n,k|n-1}^{-1}\right)$$

$$\mathbf{P}_{n|n}^{-1}\hat{\mathbf{x}}_{n|n} = \mathbf{P}_{n|n-1}^{-1}\hat{\mathbf{x}}_{n|n-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1}\hat{\mathbf{x}}_{n,k|n} - \mathbf{P}_{n,k|n-1}^{-1}\hat{\mathbf{x}}_{n,k|n-1}\right)$$

$\mathbf{y}_{0,1}$	$\mathbf{y}_{1,1}$	$\mathbf{y}_{2,1}$	$\mathbf{y}_{3,1}$

sensor 1 measurements

$\mathbf{x}_0$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$

what we want to estimate

$\mathbf{y}_{0,2}$	$\mathbf{y}_{1,2}$	$\mathbf{y}_{2,2}$	$\mathbf{y}_{3,2}$

sensor 2 measurements

sensor 1 / sensor 2

$\mathbf{P}_{0,1|0}^{-1}, \hat{\mathbf{x}}_{0,1|0}$

$\mathbf{P}_{0,2|0}^{-1}, \hat{\mathbf{x}}_{0,2|0}$

$\longmapsto$

fusion center

$$\mathbf{P}_{1,1|0} = \mathbf{A}\mathbf{P}_{0,1|0}\mathbf{A}^{\mathsf{T}} + \mathbf{Q}$$
$$\hat{\mathbf{x}}_{1,1|0} = \mathbf{A}\hat{\mathbf{x}}_{0,1|0}$$
$$\mathbf{P}_{1,2|0} = \mathbf{A}\mathbf{P}_{0,2|0}\mathbf{A}^{\mathsf{T}} + \mathbf{Q}$$
$$\hat{\mathbf{x}}_{1,2|0} = \mathbf{A}\hat{\mathbf{x}}_{0,2|0}$$
$$\mathbf{P}_{1|0} = \mathbf{A}\mathbf{P}_{0|0}\mathbf{A}^{\mathsf{T}} + \mathbf{Q}$$
$$\hat{\mathbf{x}}_{1|0} = \mathbf{A}\hat{\mathbf{x}}_{0|0}$$

Dynamic Sensor Fusion
Distributed Kalman Filtering (method 1)

say $n = 1$....what will happen ?

$$\mathbf{P}_{n|n}^{-1} = \mathbf{P}_{n|n-1}^{-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} - \mathbf{P}_{n,k|n-1}^{-1} \right)$$

$$\mathbf{P}_{n|n}^{-1} \hat{\mathbf{x}}_{n|n} = \mathbf{P}_{n|n-1}^{-1} \hat{\mathbf{x}}_{n|n-1} + \sum_{k=1}^{K} \left(\mathbf{P}_{n,k|n}^{-1} \hat{\mathbf{x}}_{n,k|n} - \mathbf{P}_{n,k|n-1}^{-1} \hat{\mathbf{x}}_{n,k|n-1} \right)$$

$\mathbf{y}_{0,1}$	$\mathbf{y}_{1,1}$	$\mathbf{y}_{2,1}$	$\mathbf{y}_{3,1}$

sensor 1 measurements

$\mathbf{x}_0$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$

what we want to estimate

$\mathbf{y}_{0,2}$	$\mathbf{y}_{1,2}$	$\mathbf{y}_{2,2}$	$\mathbf{y}_{3,2}$

sensor 2 measurements

sensor 1 / sensor 2

fusion center

Dynamic Sensor Fusion
Distributed Kalman Filtering (method 1)

say $n = 1$....what will happen ?

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$\mathbf{y}_{0,1}$	$\mathbf{y}_{1,1}$	$\mathbf{y}_{2,1}$	$\mathbf{y}_{3,1}$

sensor 1 measurements

$\mathbf{x}_0$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$

what we want to estimate

$\mathbf{y}_{0,2}$	$\mathbf{y}_{1,2}$	$\mathbf{y}_{2,2}$	$\mathbf{y}_{3,2}$

sensor 2 measurements

sensor 1 / sensor 2

$\mathbf{P}_{1,1|1}^{-1}, \hat{\mathbf{x}}_{1,1|1}$

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fusion center

Dynamic Sensor Fusion
Distributed Kalman Filtering (method 1)

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sensor 2 measurements

sensor 1 / sensor 2

fusion center

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$$\mathbf{P}_{2,1|1} = \mathbf{A}\mathbf{P}_{1,1|1}\mathbf{A}^{\mathsf{T}} + \mathbf{Q}$$

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sensor 2 measurements

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$$\begin{aligned}
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\end{aligned}$$

Dynamic Sensor Fusion
Distributed Kalman Filtering (method 2)

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sensor 2 measurements

key idea:

- the term $\mathbf{P}_{n|n-1}^{-1}\hat{\mathbf{x}}_{n|n-1}$ can be written in terms of contributions from individual sensors
- the term $\mathbf{P}_{n|n-1}^{-1}$ can be written in terms of contributions from individual sensors
- try it...

Outline

- **Star Topology**
- **General Topology**
- **One Sensor Case**
  - Model of the measurements for one sensor
  - Model of the Estimator
  - Mean Squared Error (MSE) to Chose $\mathbb{L}$
  - LMMSE Estimate
- **Combining Estimators from Many Sensors (Star Topology)**
  - Static Sensor Fusion
- **Sequential Measurements from One Sensor**
- **Sequential Measurements from Many Sensors (Star Topology)**
  - Dynamic Sensor Fusion, Centralized Setup
  - Dynamic Sensor Fusion, Centralized Setup (Drawbacks)
  - Dynamic Sensor Fusion, Distributed Kalman Filtering
- **Combining Estimators from Many Sensors (Arbitrary Topology)**
  - Static Sensor Fusion with Limited Communication Ranges

Network with Arbitrary Topology

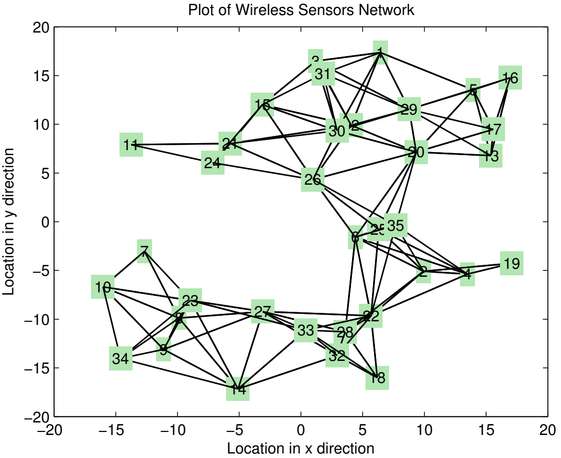

Figure: Network with a Arbitrary Topology: Solid lines indicating that there is message communication between nodes. In this network, there is no node acting as fusion center.

Network with Arbitrary Topology

- star topology: essentially a **two step procedure**
  - all the nodes transmit local estimates to a central node (**called fusion center**)
  - central node calculates and transmits the weighted sum of the local estimates back
- final outcome is a **weighted average**
- $\Rightarrow$ generalize the approach to an **arbitrary graph**
- this approaches are along the lines of **average consensus algorithms**
- **no fusion center**

Static Sensor Fusion with Limited Communication Range

example scenario:

- K nodes, each measure a **scalar** value x, measurements are noisy
- nodes are connected according to an arbitrary graph
- each node wants to calculate the average of all the scalars

$$y_k = x + v_k \ , \ k = 1, \dots, K \tag{32}$$

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important: provided the noise is iid Gaussian then the maximum likelihood **(ML)** estimate $\hat{x}$ of x is given by the average of all y_k values, i.e.,

$$\hat{x} = (1/K) \sum_{k=1}^{K} y_k = (1/K) \mathbf{1}^{\mathsf{T}} \mathbf{y} \tag{33}$$

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question: how to obtain $\hat{x}$ just by coordinating with **adjacent neighbors** (no central fusion center) ?

Static Sensor Fusion with Limited Communication Range

one way:

- iterative method, iterations $n = 0, 1, 2, \dots$
- each sensor k, during iteration 0, set $x_{0,k} = y_k$
- each sensor k implements the dynamical system

$$x_{n+1,k} = x_{n,k} + h \sum_{j \in \mathcal{N}_k} (x_{n,j} - x_{n,k}) \ , \tag{34}$$

  where $\mathcal{N}_k$ is the adjacent sensors of sensor k
- just **local communications**

Static Sensor Fusion with Limited Communication Range

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- just **local communications**
- **compact form**

$$\mathbf{x}_{n+1} = (\mathbf{I} - h\mathbf{L})\mathbf{x}_n \ , \ n = 0, 1, 2, \dots \ , \quad (35)$$

  where $\mathbf{L}$ is the **Graph Laplacian matrix ?**

Static Sensor Fusion with Limited Communication Range

question: when $n \to \infty$ do we get $(\mathbf{x}_{n+1})_k = \hat{x}$ for all $k = 1, \ldots, K$

Static Sensor Fusion with Limited Communication Range

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answer: YES

if and only if

$$(\mathbf{I} - h\mathbf{L})\mathbf{1} = \mathbf{1}, \quad \mathbf{1}^{\mathsf{T}}(\mathbf{I} - h\mathbf{L}) = \mathbf{1}^{\mathsf{T}}, \quad \rho\left((\mathbf{I} - h\mathbf{L}) - \mathbf{1}\mathbf{1}^{\mathsf{T}}\right) < 1 \tag{36}$$

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- condition 1 is true: each row sum of $\mathbf{L}$ is 0
- condition 2 is true: each column sum of $\mathbf{L}$ is 0.
- condition 3 is true: for small enough h

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the idea extends in a straightforward manner to more general models such as

$$x_{n+1,k} = x_{n,k} + h\mathbf{W}_k^{-1} \textstyle\sum_{j \in \mathcal{N}_k} (x_{n,j} - x_{n,k}) \ , \tag{37}$$