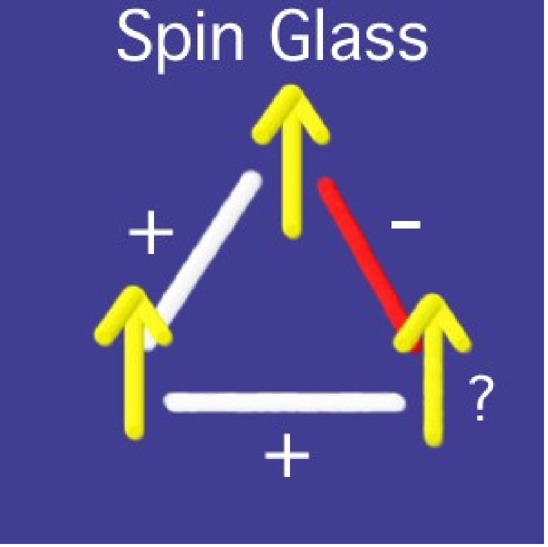

MATHEMATICAL METHODS FOR EQUILIBRIUM STATISTICAL MECHANICS OF SPIN GLASSES

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Lecture TWO

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The smooth cavity field method for the CW model.

The Sherrington-Kirkpatrick (SK) model: Generalities.

A quick overview on disordered systems: Glassy phenomenology.

A quick overview on disordered systems: Computational capabilities.

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The Mean Field Ising Model trough Interpolating Techniques

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Statistical Mechanics of Neural Networks

Lecture Notes of Course G32/NN13

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3 The Structure of the Free Energy

In this chapter we adapt the work [6] (in which a novel interpolating cavity field technique was developed for the SK model) to the mean field Ising model.

The main idea of the *cavity field* method is to look for an explicit expression of $\alpha_N(\beta) = -\beta f_N(\beta)$ upon increasing the size of the system from N particles (the cavity) to $N+1$ so that, in the limit of N that goes to infinity [25, 27]

$$\lim_{N\to\infty}(-\beta F_{N+1}(\beta)) - (-\beta F_N(\beta)) = -\beta f(\beta) \tag{23}$$

because the existence of the thermodynamic limit (Sect. 2.2) implies only vanishing correction of the free energy density.

Note Strictly speaking the limit does exist surely just in the Cesàro sense [23] (Cesàro limits are employed when analyzing sequences which can oscillate and do not converge, i.e. the Liebnitz series converges to zero in the Cesàro sense [36]) but this level of mathematical rigor will not be presented along the paper.

3.1 Interpolating Cavity Field

As we will see, the interpolating technique can be very naturally implemented in the cavity method; let us consider the partition function of a system made by $N+1$ spins:

$$Z_{N+1}(\beta) = \sum_{\sigma} e^{-\beta H_{N+1}(\sigma)}$$
$$= \sum_{\sigma_{N+1}=\pm 1} \sum_{\sigma} e^{\frac{\beta}{N+1}\sum_{1\le i<j\le N}\sigma_i\sigma_j} e^{\frac{\beta}{N+1}\sum_{1\le i\le N}\sigma_i\sigma_{N+1}}. \quad (24)$$

With the gauge transformation $\sigma_i \to \sigma_i\sigma_{N+1}$, which, of course, is a symmetry of the Hamiltonian, we get

$$Z_{N+1}(\beta) = 2Z_N(\beta^*)\tilde{\omega}(e^{\frac{\beta}{N+1}\sum_{1\le i\le N}\sigma_i}) \quad (25)$$

where $\tilde{\omega}$ is the Boltzmann state at the inverse temperature $\beta^* = \beta\frac{N}{N+1}$ (note that in the thermodynamic limit the shifted temperature converges to the real one $\beta^* \to \beta$). Let us reverse the temperature shift and apply the logarithm to both the sides of (25) to obtain

$$\ln Z_{N+1}\left(\beta\frac{N+1}{N}\right) = \ln 2 + \ln Z_N(\beta) + \ln\omega_N(e^{\frac{\beta}{N}\sum_{1\le i\le N}\sigma_i}). \quad (26)$$

Equation (26) tell us that via the third term of its r.h.s. we can bridge an Ising system with N particles at an inverse temperature β to an Ising system with $N+1$ particles at a shifted inverse temperature $\beta^* = \beta(N+1)/N$. Focusing on such a term let us make the following definitions.

Definition 1 We define an extended partition function $Z_N(\beta, t)$ as

$$Z_N(\beta, t) = \sum_\sigma e^{-\beta H_N(\sigma)} e^{\frac{t}{N}\sum_{1\le i\le N}\sigma_i}. \tag{27}$$

Note that the above partition function, at $t = \beta$, turns out to be, via the global gauge symmetry $\sigma_i \to \sigma_i\sigma_{N+1}$, a partition function for a system of $N+1$ spins at a shifted temperature β^* apart a constant term. On the same line

Definition 2 We define the generalized Boltzmann state $\langle\,\rangle_t$ as

$$\langle F(\sigma)\rangle_t = \frac{\langle F(\sigma) e^{\frac{t}{N}\sum_{1\le i\le N}\sigma_i}\rangle}{\langle e^{\frac{t}{N}\sum_{1\le i\le N}\sigma_i}\rangle}, \tag{28}$$

$F(\sigma)$ being a generic function of the spins.

Definition 3 Related to the Boltzmann state $\langle \rangle$ we define the cavity function $\Psi(\beta, t) = \lim_{N\to\infty} \Psi_N(\beta, t)$ as

$$\Psi(\beta, t) = \lim_{N\to\infty} \Psi_N(\beta, t) = \lim_{N\to\infty} \ln\langle e^{\frac{t}{N}\sum_{1\le i\le N}\sigma_i}\rangle. \tag{29}$$

It will appear clear, while reading the paper, when we deal with the finite-N cavity function and when with its thermodynamic limit.

Definition 4 We define respectively as fillable and filled monomials the odd and even momenta of the magnetization weighted by the extended Boltzmann measure such that

- $\langle m_N^{2n+1}\rangle_t$ with $n \in \mathbb{N}$ is fillable
- $\langle m_N^{2n}\rangle_t$ with $n \in \mathbb{N}$ is filled

Proposition 1 *The cavity function $\Psi(\beta, t)$ is the generating function of the centered momenta of the magnetization, examples of which are*

$$\frac{\partial \Psi_N(\beta, t)}{\partial t} = \langle m_N\rangle_t, \tag{30}$$

$$\frac{\partial^2 \Psi_N(\beta, t)}{\partial t^2} = \langle m_N^2\rangle_t - \langle m_N\rangle_t^2. \tag{31}$$

Theorem 2 *The following relation holds in the thermodynamic limit*:

$$\alpha(\beta) = \ln 2 + \Psi(\beta, t = \beta) - \beta \frac{\partial \alpha(\beta)}{\partial \beta}. \tag{32}$$

Proof Let us consider again the partition function of a system made up by $(N+1)$ spins and point out with β the true temperature and with $\beta^* = \beta(1 + N^{-1})$ the shifted one:

$$Z_{N+1}(\beta) = \sum_{\sigma_{N+1}} e^{\frac{\beta}{\sqrt{N+1}} \sum_{1 \le i < j \le N+1} \sigma_i \sigma_j}$$

$$= 2 \sum_{\sigma_N} e^{\frac{\beta^*}{\sqrt{N}} \sum_{1 \le i < j \le N} \sigma_i \sigma_j} e^{\frac{\beta}{\sqrt{N+1}} \sum_{1 \le i \le N} \sigma_i}. \tag{33}$$

Typo!

Now we multiply and divide by $Z_N(\beta^*)$ the right hand side of (33), then we take the logarithm on both sides and subtract from every member the quantity $\ln Z_{N+1}(\beta^*)$; expanding $\ln Z_{N+1}(\beta)$ around $\beta = \beta^*$ as

$$\ln Z_{N+1}(\beta) - \ln Z_{N+1}(\beta^*) = (\beta - \beta^*) \partial_{\beta^*} \ln Z_{N+1}(\beta^*) + O((\beta - \beta^*)^2) \tag{34}$$

with

$$\beta - \beta^* = \beta^* \left(\sqrt{\frac{N+1}{N}} - 1 \right) = \frac{\beta^*}{2N} + O(N^{-1}) \tag{35}$$

we substitute β with β^* inside the state ω and neglecting corrections $O(N^{-1})$ we have:

$$\begin{aligned} &\ln Z_{N+1}(\beta^*) + (\beta - \beta^*)\partial_{\beta^*} \ln Z_{N+1}(\beta^*) \\ &= \ln 2 + \ln Z_N(\beta^*) + \ln \omega_{N,\beta^*}(e^{\frac{\beta}{\sqrt{N+1}} \sum_{1<i<N} \sigma_i}) + O(N^{-1}), \end{aligned} \tag{36}$$

where, with the symbol ω_{N,β^*} we stressed that the temperature inside the Boltzmann average is the shifted one. Using the variable $\alpha(\beta^*)$ and renaming $\beta^* \to \beta$ in the thermodynamic limit we get:

$$\alpha(\beta) + \beta \frac{d\alpha(\beta)}{d\beta} = \ln 2 + \Psi(\beta, t = \beta) \tag{37}$$

and this is the thesis of the theorem. □

3.3 Saturability and Gauge-Invariance

The next step is to motivate why we introduced the whole machinery: The first reason we are going to show are peculiar properties of both the filled and the fillable monomials (see Definition 4). In the thermodynamic limit, the first class do not depend on the perturbation induced by the cavity field and, at $t = \beta$, the latter (via the $\sigma_i \to \sigma_i \sigma_{N+1}$ symmetry) is projected into the first class. The second reason is that the free energy can be expanded via these monomials, so a good control of them means a good knowledge of the thermodynamic of the system.

Theorem 3 *In the $N \to \infty$ limit the averages $\langle m_N^{2n} \rangle$ of the filled monomials are t-independent for almost all values of β, such that*

$$\lim_{N\to\infty} \partial_t \langle m_N^{2n} \rangle_t = 0.$$

Proof Without loss of generality we will prove the theorem in the simplest case (for $\langle m_N^2 \rangle$); it will appear immediately clear how to generalize the proof to higher order monomials. Let us write the cavity function as

$$\Psi_N(\beta, t) = \ln Z_N(\beta, t) - \ln Z_N(\beta) \tag{38}$$

and derive it with respect to β:

$$\Psi(\beta, t) = \lim_{N\to\infty} \Psi_N(\beta, t) = \lim_{N\to\infty} \ln\langle e^{\frac{t}{N}\sum_{1\le i\le N}\sigma_i}\rangle.$$

$$\frac{\partial \Psi_N(\beta, t)}{\partial \beta} = \frac{N}{2}(\langle m_N^2 \rangle - \langle m_N^2 \rangle_t). \tag{39}$$

We can introduce an auxiliary function $\Upsilon_N(\beta, t) = (\langle m_N^2 \rangle - \langle m_N^2 \rangle_t)$ such that:

$$\Upsilon_N(\beta, t) = \frac{2}{N}\partial_\beta \Psi_N(\beta, t) \tag{40}$$

and integrate it in a generic interval $[\beta_1, \beta_2]$:

$$\int_{\beta_1}^{\beta_2} \Upsilon_N(\beta, t) d\beta^2 = \frac{4}{N}[\Psi_N(\beta_2, t) - \Psi_N(\beta_1, t)]. \tag{41}$$

Now we must control $\Psi_N(\beta, t)$ in the $N \to \infty$ limit; the simplest way is to look at its t-streaming $\partial_t \Psi_N(\beta, t) = \langle m_N \rangle_t$ such the N-dependence is just taken into account by the Boltzmann factor inside the averages and, as $\langle m_N \rangle_t \in [-1, 1]$, in the thermodynamic limit $\Psi(\beta, t)$ remains bounded and the second member of (41) goes to zero such that, $\forall\ [\beta_1, \beta_2]$, $\Upsilon_N(\beta, t)$ converges to zero implying $\langle m_N^2 \rangle_t \to \langle m_N^2 \rangle$. □

Remark 3 A consequence of this property, in the spin glass theory, turns out to be the stochastic stability of a large class of overlap polynomials [16, 41].

The next theorem is crucial for this section, so, for the sake of simplicity, we split it in two part: at first we prove the following lemma than it will make us able to proof the core of the theorem itself which will be showed immediately after. For a clearer statement of the lemma we take the freedom of pasting the volume dependence of the averages as a subscript close to the perturbing tuning parameter t.

Lemma 1 *Let $\langle\rangle_N$ and $\langle\rangle_{N,t}$ be the states defined, on a system of N spins, respectively by the canonical partition function $Z_N(\beta)$ and by the extended one $Z_N(\beta,t)$; if we consider the ensemble of indexes $\{i_1,\dots,i_r\}$ with $r\in[1,N]$, then for $t=\beta$, where the two measures become comparable, thanks to the global gauge symmetry (i.e. the substitution $\sigma_i\to\sigma_i\sigma_{N+1}$) the following relation holds*

$$\omega_{N,t=\beta}(\sigma_{i_1}\cdots\sigma_{i_r})=\omega_{N+1}(\sigma_{i_1}\cdots\sigma_{i_r}\sigma_{N+1}^r)+O\left(\frac{1}{N}\right) \tag{42}$$

where r is an exponent, so if r is even $\sigma_{N+1}^r=1$, while if it is odd $\sigma_{N+1}^r=\sigma_{N+1}$.

Proof Let us write $\omega_{N,t}$ for $t=\beta$, defining for the sake of simplicity $\pi = \sigma_{i_1}\cdots\sigma_{i_r}$:

$$\omega_{N,t=\beta}(\pi) = \left[\sum_{\sigma} \frac{1}{Z_N(\beta)} e^{\frac{\beta}{\sqrt{N}}\sum_{1\le i<j\le N}\sigma_i\sigma_j + \frac{\beta}{\sqrt{N}}\sum_i \sigma_i}\pi\right]. \quad (43)$$

Introducing first a sum over σ_{N+1} at the numerator and at the denominator, (which is the same as multiply and divide for 2 because there is still no dependence to σ_{N+1}) and making the transformation $\sigma_i \to \sigma_i\sigma_{N+1}$, the variable σ_{N+1} appears at the numerator and it is possible to build the status at $N+1$ particles with the little temperature shift which vanishes in the thermodynamic limit:

$$\omega_{N,t=\beta}(\pi) = \omega_{N+1}(\pi\sigma^r_{N+1}) + O\left(\frac{1}{N}\right). \quad (44)$$

□

Using this lemma we are able to proof the following

Theorem 4 *Let $\langle M\rangle_t$ be a fillable monomial of the magnetization, (this means that $\langle mM\rangle$ is filled). We have:*

$$\lim_{N\to\infty}\lim_{t\to\beta}\langle M\rangle_t = \langle mM\rangle. \quad (45)$$

Proof The proof is a straightforward application of Lemma 1. □

Usually it is much simpler to evaluate the internal energy than the free energy because there is no contribution by the entropy, which, especially in complex system, can make things much harder; consequently if we learn how to extrapolate information from the cavity function, which is deeply related to the entropy, we can obtain information for the free energy. To fulfill this task we state the following theorem.

Theorem 5 *When a generic well defined function of the spins $F(\sigma)$ is considered, the following streaming equation holds*:

$$\frac{\partial \langle F_N(\sigma)\rangle_t}{\partial t} = \langle F_N(\sigma) m_N\rangle_t - \langle F_N(\sigma)\rangle_t \langle m_N\rangle_t. \tag{46}$$

Proof The proof is straightforward and can be obtained by simple derivation:

$$\begin{aligned}
\frac{\partial \langle F_N(\sigma)\rangle_t}{\partial t} &= \partial_t \frac{\sum_\sigma F_N(\sigma) e^{-\beta H_N(\sigma)} e^{\frac{t}{N}\sum_{1<i<N}\sigma_i}}{\sum_\sigma e^{-\beta H_N(\sigma)} e^{\frac{t}{N}\sum_{1<i<N}\sigma_i}} \\
&= \left(\frac{\sum_\sigma F_N(\sigma)\frac{1}{N}\sum_{1<i<N}\sigma_i e^{-\beta H_N(\sigma)} e^{\frac{t}{N}\sum_{1<i<N}\sigma_i}}{\sum_\sigma e^{-\beta H_N(\sigma)}}\right) \\
&\quad - \left(\frac{\sum_\sigma F_N(\sigma) e^{-\beta H_N(\sigma)} e^{\frac{t}{N}\sum_{1<i<N}\sigma_i}}{\sum_\sigma e^{-\beta H_N(\sigma)}}\right) \\
&\quad \times \left(\frac{\sum_\sigma \frac{1}{N}\sum_{1<i<N}\sigma_i e^{-\beta H_N(\sigma)} e^{\frac{t}{N}\sum_{1<i<N}\sigma_i}}{\sum_\sigma e^{-\beta H_N(\sigma)}}\right) \\
&= \langle F_N(\sigma) m_N\rangle_t - \langle F_N(\sigma)\rangle_t \langle m_N\rangle_t. \qquad \square
\end{aligned}$$

We now want to expand the cavity function via filled monomials of the magnetization by applying the streaming equation (46) directly to its derivative, thanks to (30). It is immediate to find that the streaming of $\langle m_N \rangle_t$ obeys the following differential equation

$$\partial_t \langle m_N \rangle_t = \langle m_N^2 \rangle_t - \langle m_N \rangle_t^2 \tag{47}$$

which, thanks to Theorem 4, becomes trivial in the thermodynamic limit. In fact, calling $m = \lim_{N\to\infty} m_N$ and skipping the subscript t on $\lim_{N\to\infty} \langle m_N^2 \rangle_t = \langle m^2 \rangle$ we obtain

$$\frac{1}{\langle m^2 \rangle} \partial_t \langle m \rangle_t = 1 - \left(\frac{\langle m \rangle_t^2}{\langle m^2 \rangle} \right)$$

which is easily solved by splitting the variables and the solution is $\int \frac{1}{1-x^2} dx = \frac{1}{2} \log \left| \frac{1+x}{1-x} \right|$

$$\langle m \rangle_t = \sqrt{\langle m^2 \rangle} \tanh(\sqrt{\langle m^2 \rangle} t). \tag{48}$$

Once evaluated (48) by using the gauge at $t = \beta$ (i.e. $\langle m \rangle_{t=\beta} = \langle m^2 \rangle$) we get

$$\sqrt{\langle m^2 \rangle} = \tanh(\beta \sqrt{\langle m^2 \rangle}) \tag{49}$$

which is the well known self-consistency equation for the Ising-model.

3.5 The Free Energy Expansion

From (48) it is possible to obtain an explicit expression for the cavity function to plug into (32) solving for the free energy. In fact we have

$$\lim_{N\to\infty} \Psi_N(\beta,t) = \lim_{N\to\infty} \int_0^t dt' \langle m_N \rangle_{t'} = \int_0^t dt' \sqrt{\langle m^2 \rangle} \tanh(\sqrt{\langle m^2 \rangle} t') \tag{50}$$

from which is immediate to solve for $\Psi(\beta,t)$:

$$\Psi(\beta,t) = \ln\cosh(\sqrt{\langle m^2 \rangle} t). \tag{51}$$

The last term still missing to fulfill the expression of the free energy via (32), which is immediate to obtain, is the internal energy.

Proposition 2 *The internal energy of the Ising model is*

$$\beta \frac{d\alpha_N(\beta)}{d\beta} = \frac{\beta}{2} \langle m_N^2 \rangle. \tag{52}$$

Proof The proof is straightforward and can be obtained by direct calculation on the same line of the previous proofs. □

Pasting all together we have

Proposition 3 *The free energy of the Ising model is*

$$\alpha(\beta) = \ln 2 + \ln\cosh(\beta\sqrt{\langle m^2\rangle}) - \frac{\beta}{2}(\sqrt{\langle m^2\rangle})^2. \tag{53}$$

4 The Phase Transition

4.1 Breaking Commutativity of Infinite Volume Against Vanishing Perturbation Limits

The motivation of this section can be found, always in the context of spin glasses in [9].

Let us move one step backward and consider (53) at finite N. The receipt to obtain the expression of the free energy via the filled monomial is to perform at first the $N \to \infty$ limit to saturate the fillable term and then the $t \to \beta$ limit to free the measure from the perturbation (making it works as a cavity field). So in other words $\alpha(\beta) = \lim_{t\to\beta}\lim_{N\to\infty}\alpha_N(\beta,t)$. But what if we exchange the limits such that $\alpha^*(\beta) = \lim_{N\to\infty}\lim_{t\to\beta}\alpha_N(\beta,t)$?

Simply, thanks to the gauge invariance $\lim_{N\to\infty}\lim_{t\to\beta}\langle m_N\rangle = 0$ implying $\Psi(\beta,t) = 0$, defining the high temperature expression for $\alpha^*(\beta)$, so

$$\alpha(\beta) = \lim_{t\to\beta}\lim_{N\to\infty}\alpha_N(\beta,t) \neq \lim_{N\to\infty}\lim_{t\to\beta}\alpha_N(\beta,t) = \alpha^*(\beta). \tag{54}$$

Alternatively one can solve (47) for the variable $\langle\xi_N(\sigma)\rangle_t$ by sending first $N \to \infty$ and check that these fluctuations scale accordingly to the paragraph after (5).

4.2 Critical Behavior: Scaling Laws

Critical exponents are needed to characterize singularities of the theory at the critical point and, for us, this information is encoded in the behavior of the order parameter $\sqrt{\langle m^2\rangle}$.

Assuming for the moment that $\beta_c = 1$ (where β_c stands for the critical point in temperature), close to criticality, we take the freedom of writing $G(\beta) \sim G_0 \cdot (\beta - 1)^{\gamma}$, where the symbol $\sim$ has the meaning that the term at the second member is the dominant but there are corrections of order higher than τ^{γ}.

The standard way to look at the scaling of the order parameter is by expanding the hyperbolic tangent around $\sqrt{\langle m^2\rangle} \sim 0$ obtaining

$$\sqrt{\langle m^2\rangle} = \tanh(\beta\sqrt{\langle m^2\rangle}) \sim \beta\sqrt{\langle m^2\rangle} - \frac{(\beta\sqrt{\langle m^2\rangle})^3}{3} \tag{55}$$

by which one gets

$$\sqrt{\langle m^2\rangle}(1-\beta) + \frac{1}{3}(\beta(\sqrt{\langle m^2\rangle})^3) \sim 0. \tag{56}$$

The first solution of (56) is $\sqrt{\langle m^2 \rangle} = 0$ (which is also the only solution in the ergodic phase) while the other two solutions can be obtained by solving

$$(\sqrt{\langle m^2 \rangle})^2 \sim \frac{(\beta - 1)^3}{\beta^3} \sim 3\left(1 - \frac{1}{\beta}\right) \tag{57}$$

close to the critical point, obtaining

$$\sqrt{\langle m^2 \rangle} \sim (\beta - 1)^{\frac{1}{2}} \tag{58}$$

which gives as the critical exponent $\gamma = 1/2$.

4.3 Self-Averaging Properties

As a sideline, to try and make the work as close as possible to a guide for more complex models, it is possible to derive the "locking" of the order parameter, which, in other context (i.e. spin glasses) is found as a set of equations called Ghirlanda-Guerra [20] and Aizenman-Contucci [3], while in simpler systems as the one we are analyzing, not surprisingly [16], do coincide with just one kind of self-averaging.

The idea we follow [6–8] is deriving filled monomial with respect to the interpolating parameter, remembering that, in the thermodynamic limit, they do not depend on such a parameter end evaluating the "fillable" result (which do depends on t) at $t = \beta$ to free the measure from the perturbing cavity field.

Proposition 4 *The self-averaging properties, consequence of the invariance of filled monomials with respect the perturbing field, hold in the thermodynamic limit; an example being*

$$0 = \lim_{N\to\infty} \partial_t \langle m_N^2 \rangle = \langle m^3 \rangle_t - \langle m^2 \rangle \langle m \rangle_t = \langle m^4 \rangle - \langle m^2 \rangle^2. \quad (65)$$

Remark 6 The self-averaging property of the order parameter is a consequence of self-averaging of the internal energy

$$\lim_{N\to\infty}(\langle E_N\rangle^2-\langle E_N^2\rangle)=0 \quad \Rightarrow \quad (\langle m^2\rangle^2-\langle m^4\rangle)=0.$$

Note In this system without disorder the AC relations and the GG identities do coincide because of the absence of the external average over the noise, which introduce different kinds of self-averaging as discussed for instance in [18].

A less known alternative, richer of surprises, emerges again when investigating the cavity function. Of course in simple system such investigation will not tell us much more than what showed so far, but, remembering we want to show a working method more than the results themselves it offers for this particular system, we want to explore this last variant.

Remembering Theorem 4 and Proposition 3 let us rewrite the free energy according to

$$\alpha(\beta) = \ln 2 + \ln\cosh(\sqrt{\langle m \rangle_t}\, t)|_{t=\beta} - \frac{\beta}{2}\sqrt{\langle m^2 \rangle} \tag{66}$$

and emphasize that the total derivative with respect to β is

$$\frac{d\alpha(\beta)}{d\beta} = \frac{\partial\alpha(\beta)}{\partial\beta} + \frac{\partial\alpha(\beta)}{\partial\sqrt{\langle m^2 \rangle}}\frac{\partial\sqrt{\langle m^2 \rangle}}{\partial\beta}, \tag{67}$$

while, from the general law of thermodynamics [42], we know the total derivative of the free energy with respect to β is the internal energy

$$\frac{d\alpha(\beta)}{d\beta} = \frac{1}{2}(\sqrt{\langle m^2 \rangle})^2. \tag{68}$$

With this preamble let us move evaluating the partial derivative of the free energy still with respect β:

$$\frac{\partial \alpha(\beta)}{\partial \beta} = -\frac{1}{2}(\sqrt{\langle m^2 \rangle})^2 + (\sqrt{\langle m \rangle_t} \tanh(\sqrt{\langle m \rangle_t} t))|_{t=\beta}$$
$$= -\frac{1}{2}(\sqrt{\langle m^2 \rangle})^2 + (\sqrt{\langle m^2 \rangle} \tanh(\sqrt{\langle m^2 \rangle} \beta)$$

which thanks to self-consistency for the order parameter (49) becomes

$$-\frac{1}{2}(\sqrt{\langle m^2 \rangle})^2 + (\sqrt{\langle m^2 \rangle})^2 = \frac{1}{2}(\sqrt{\langle m^2 \rangle})^2 \qquad (69)$$

hence

$$\frac{\partial \alpha(\beta)}{\partial \sqrt{\langle m^2 \rangle}} \frac{\partial \sqrt{\langle m^2 \rangle}}{\partial \beta} = 0. \qquad (70)$$

Let us split the evaluation of (70) in two terms A, B (such that the equation reduces to $AB = 0$) by defining and evaluating

$$A = \frac{\partial\alpha(\beta)}{\partial\sqrt{\langle m^2\rangle}} = \beta(\sqrt{\langle m^2\rangle} - \tanh(\beta\sqrt{\langle m^2\rangle})), \tag{71}$$

$$B = \frac{\partial\sqrt{\langle m^2\rangle}}{\partial\beta} = \frac{N}{4\sqrt{\langle m^2\rangle}}(\sqrt{\langle m^4\rangle} - (\sqrt{\langle m^2\rangle})^2). \tag{72}$$

Putting together the results $AB = 0$ we obtain

$$\beta(\sqrt{\langle m^2\rangle} - \tanh(\beta\sqrt{\langle m^2\rangle}))\frac{N}{4\sqrt{\langle m^2\rangle}}(\sqrt{\langle m^4\rangle} - (\sqrt{\langle m^2\rangle})^2) = 0. \tag{73}$$

This equation acts as a bound and, thought in terms of the expression (70), has a vague variational taste. As in simple system it does not tell us much more than that the product of self-consistency and self-averaging goes to zero faster than N^{-1}, in complex system has a key role both in defining the locking of the order parameters [6] as in controlling the system at criticality [10]. Furthermore in such equation the two key ingredient for the behavior of the system, i.e. self-consistency and self-averaging, appear together as a whole.

1.1 Stochastic Local Field Alignment

Parallel Dynamics. The microscopic laws governing the parallel evolution of a system of N Ising spin neurons $\sigma_i \in \{-1, 1\}$ are defined as a stochastic alignment to local fields $h_i(\boldsymbol{\sigma})$. These fields represent the post-synaptic potentials of the neurons and are assumed to depend linearly on the instantaneous neuron states:

$$\sigma_i(t+1) = \mathrm{sgn}\left[\tanh\left[\beta h_i(\boldsymbol{\sigma}(t))\right] + \eta_i(t)\right] \tag{1.1}$$

$$h_i(\boldsymbol{\sigma}(t)) \equiv \sum_{j=1}^{N} J_{ij}\sigma_j(t) + \theta_i(t) \tag{1.2}$$

The stochasticity is in the independent random numbers $\eta_i(t)$ (representing threshold noise), which are distributed uniformly over the interval $[-1, 1]$. The parameter β controls the impact of this noise on the states $\sigma_i(t+1)$. For $\beta = \infty$ the random numbers cannot influence the system state and the process becomes deterministic: $\sigma_i(t+1) = \mathrm{sgn}[h_i(\boldsymbol{\sigma}(t))]$. The opposite extreme is choosing $\beta = 0$, in which case the system evolution becomes fully random. The external fields $\theta_i(t)$ represent neural thresholds and/or external stimuli. The specific choice $tanh$ for the non-linearity in definition (1.1) is only relevant for the special case of symmetric interactions. There it allows us to identify the sequential version of this stochastic dynamics as a Glauber (1963) dynamics with respect to the standard Ising spin Hamiltonian, and as a consequence apply standard equilibrium statistical mechanics.

The microscopic equations (1.1) can be transformed directly into equations for the evolution of the microscopic state probability $p_t(\boldsymbol{\sigma})$, with $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N)$. If $\boldsymbol{\sigma}(t)$ is given we find

$$p_{t+1}(\boldsymbol{\sigma}) = \prod_{i=1}^{N} \frac{1}{2}\left[1 + \sigma_i \tanh[\beta h_i(\boldsymbol{\sigma}(t))]\right] = \prod_{i=1}^{N} \frac{e^{\beta\sigma_i h_i(\boldsymbol{\sigma}(t))}}{2\cosh[\beta h_i(\boldsymbol{\sigma}(t))]}$$

If, instead of $\boldsymbol{\sigma}(t)$, the probability distribution $p_t(\boldsymbol{\sigma})$ is given, the above expression generalises to the corresponding average over the states at time t:

$$p_{t+1}(\boldsymbol{\sigma}) = \sum_{\boldsymbol{\sigma}'} W\left[\boldsymbol{\sigma};\boldsymbol{\sigma}'\right] p_t(\boldsymbol{\sigma}') \tag{1.3}$$

$$W\left[\boldsymbol{\sigma};\boldsymbol{\sigma}'\right] \equiv \prod_{i=1}^{N} \frac{e^{\beta\sigma_i h_i(\boldsymbol{\sigma}')}}{2\cosh[\beta h_i(\boldsymbol{\sigma}')]} \tag{1.4}$$

which is the Markov equation corresponding to the parallel process (1.1).

Detailed Balance. The results obtained above suggest that symmetric systems, where $J_{ij} = J_{ji}$ for all (ij), represent a special class. We now show how interaction symmetry is closely related to the detailed balance property, and derive a number of consequences. A Markov process of the form (1.3,1.7), i.e.

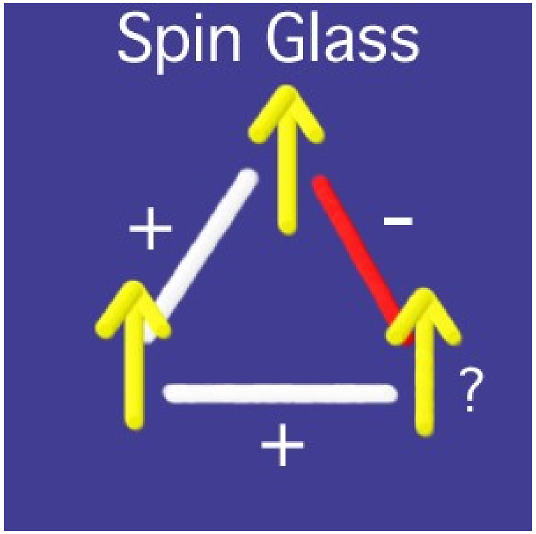
$$p_{t+1}(\boldsymbol{\sigma}) = \sum_{\boldsymbol{\sigma}'} W\left[\boldsymbol{\sigma};\boldsymbol{\sigma}'\right] p_t(\boldsymbol{\sigma}') \tag{1.17}$$

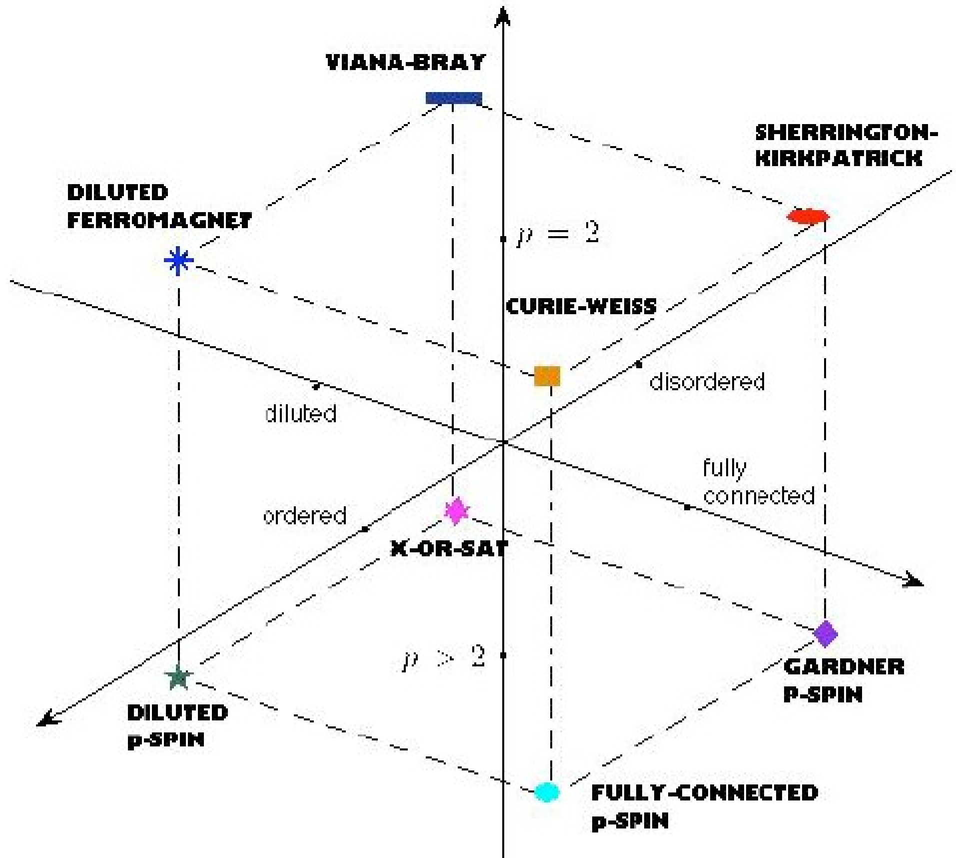
$$W\left[\boldsymbol{\sigma};\boldsymbol{\sigma}'\right] \in [0,1] \qquad\qquad \sum_{\boldsymbol{\sigma}} W\left[\boldsymbol{\sigma};\boldsymbol{\sigma}'\right] = 1$$

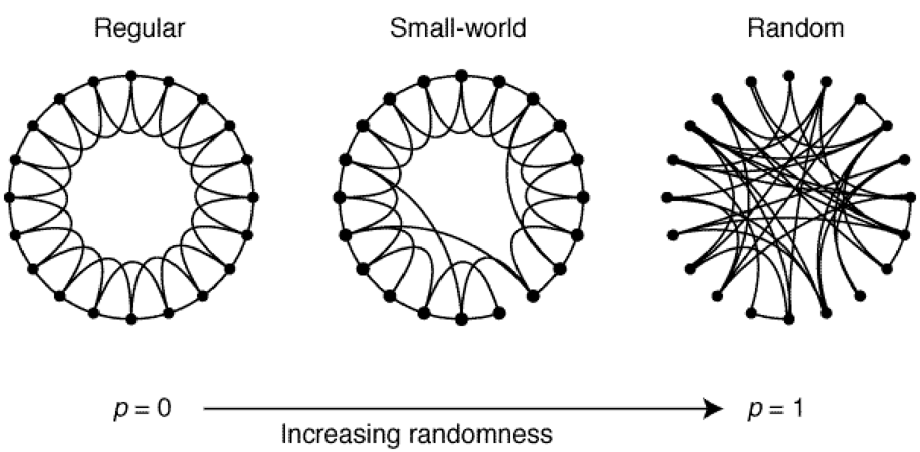
(where the conditions on the transition matrix ensure a probabilistic interpretation of $p_t(\boldsymbol{\sigma})$) is said to obey detailed balance if there exists a stationary solution $p_\infty(\boldsymbol{\sigma})$ of (1.17) with the property:

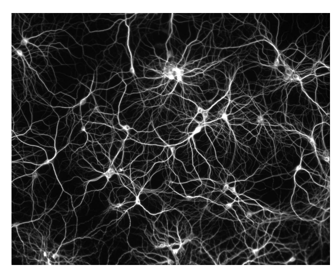
$$W\left[\boldsymbol{\sigma};\boldsymbol{\sigma}'\right] p_\infty(\boldsymbol{\sigma}') = W\left[\boldsymbol{\sigma}';\boldsymbol{\sigma}\right] p_\infty(\boldsymbol{\sigma}) \qquad \text{for all } \boldsymbol{\sigma},\boldsymbol{\sigma}' \tag{1.18}$$

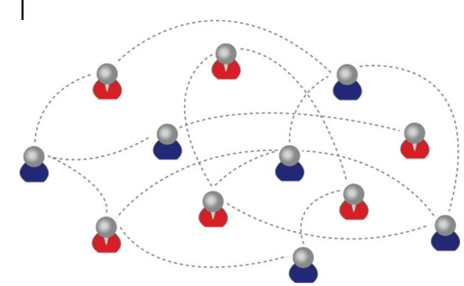
Let us start describing the “complex” phenomenology: Generalities on SK model and disordered systems.







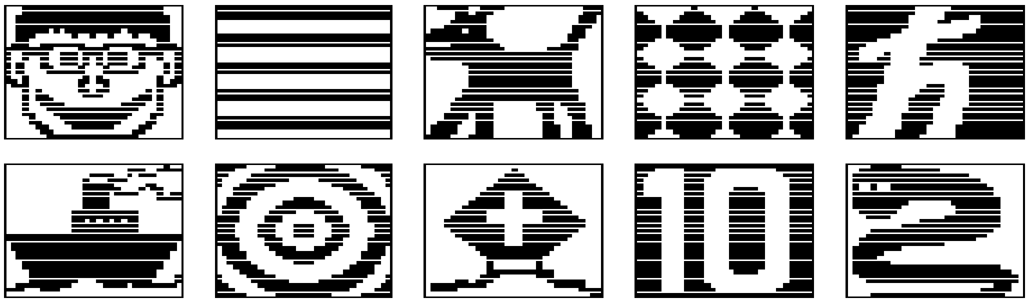




Recurrent Networks in Equilibrium

The Hopfield model is obtained by generalising the recipe (2.1) to the case of having an arbitrary number p of patterns:

$$J_{ij} = \frac{1}{N}\sum_{\mu=1}^{p} \xi_i^{\mu}\xi_j^{\mu}, \qquad \theta_i = 0$$



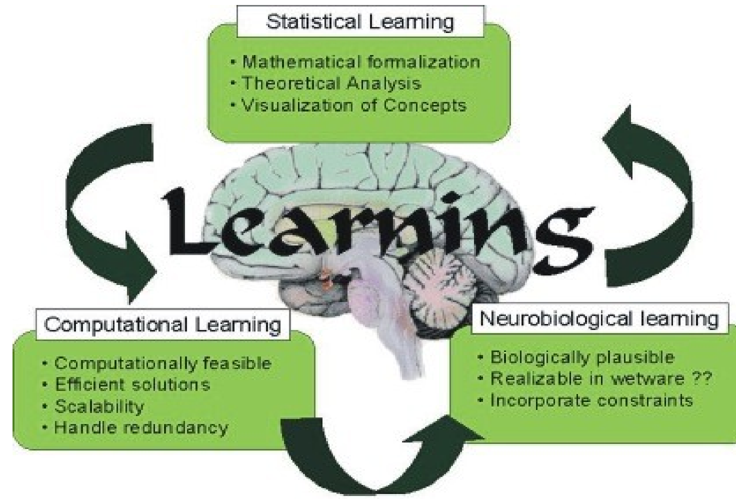


Figure 2.3: Information storage with the Hopfield model: $p = 10$ patterns represented as specific microscopic spin configurations in an $N = 841$ network.