Polarized and unpolarised transverse waves

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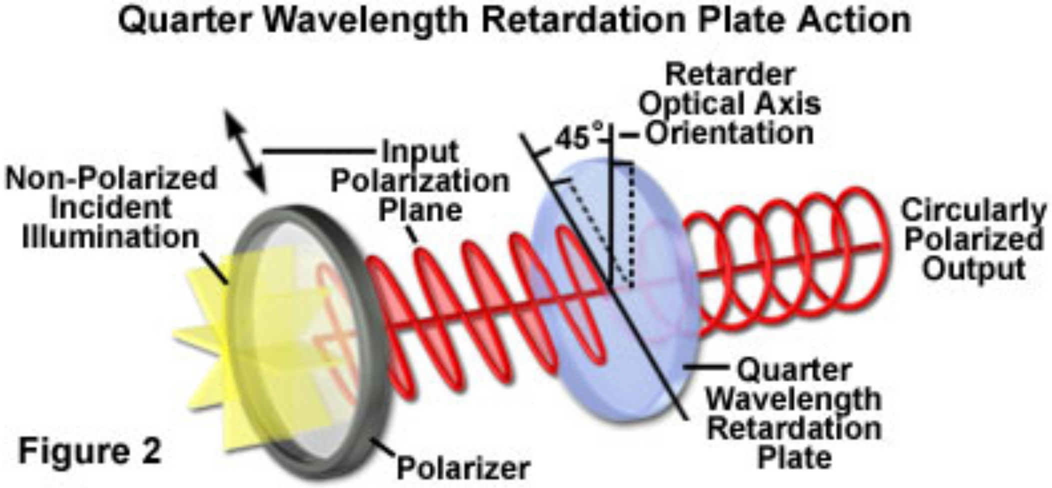
Outline

- The quarter wave plate
- Set up coordinate system suitable for transverse waves
- **Jones calculus**; matrix formulation of how wave polarization changes when passing through polarizing component
  - Examples: linear polarizer, quarter wave plate, Faraday rotation
- Statistical representation of incoherent/unpolarized waves
  - Stokes vector and polarization tensor
    - Poincare sphere
- **Muller calculus**; matrix formulation for the transmission of partially polarized waves

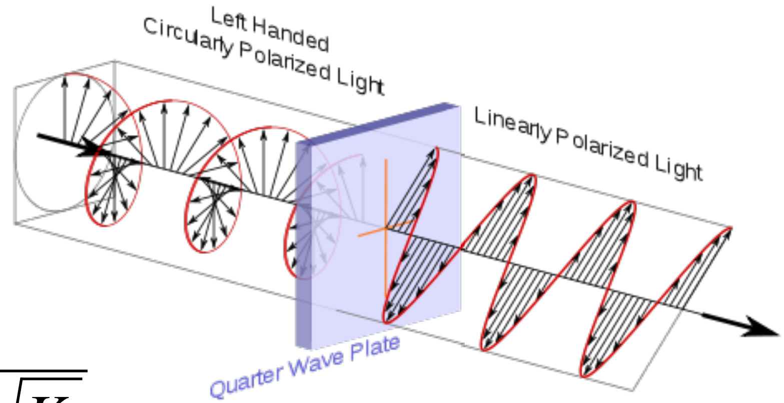


Modifying wave polarization in a quarter wave plate (1)

- Last lecture we noted that in birefringent crystals:
  - there are two modes: O-mode and X-mode

$$\begin{cases} n_O{}^2 = K_\perp \\ n_X{}^2 = \dfrac{K_\perp K_\parallel}{K_\perp \sin^2\theta + K_\parallel \cos^2\theta} \end{cases} \qquad \begin{cases} \mathbf{e}_O(\mathbf{k}) = (0\ ,1\ ,0) \\ \mathbf{e}_X(\mathbf{k}) \propto (K_\parallel \cos\theta\ ,\ 0\ ,\ K_\perp \sin\theta) \end{cases}$$

  - thus if $K_\perp > K_\parallel$ then $n_O \geq n_X$
    the O-mode has larger phase velocity



- Next describe Quarter wave plates
  - uniaxial crystal; normal in z-direction
  - length L in the x-direction: $L = \dfrac{c}{\omega}\dfrac{\pi/2}{\sqrt{K_\parallel} - \sqrt{K_\perp}}$
  - Let a wave travel in the x-direction, then k is in the x-direction and $\theta=\pi/2$

$$\begin{cases} n_O{}^2 = K_\perp \\ n_X{}^2 = K_\parallel \end{cases} \qquad \begin{cases} \mathbf{e}_O(\mathbf{k}) = (0\ ,1\ ,0) \\ \mathbf{e}_X(\mathbf{k}) = (0\ ,\ 0\ ,1) \end{cases}$$

Modifying wave polarization in a quarter wave plate (2)

- Plane wave ansats has to match dispersion relation
  - when the wave entre the crystal it will move slow, this corresponds to a change in wave length, or k

$$k_O = \frac{\omega n_O}{c} = \frac{\omega}{c}\sqrt{K_\perp} \ , \ k_X = \frac{\omega n_X}{c} = \frac{\omega}{c}\sqrt{K_\parallel}$$

  - since the O- and X-mode travel at different speeds we write

$$\mathbf{E}(t,x) = \Re\left\{\mathbf{e}_O E_O \exp(ik_O x - i\omega t) + \mathbf{e}_X E_X \exp(ik_X x - i\omega t)\right\}$$

- Assume as initial condition a linearly polarized wave

$$\mathbf{E} = \begin{bmatrix}1 & 1 & 0\end{bmatrix} = (\mathbf{e}_O + \mathbf{e}_X) \Rightarrow E_O = E_X = 1$$

$$\Rightarrow \mathbf{E}(t,x) = \Re\left\{\mathbf{e}_O \exp(ik_O x - i\omega t) + \mathbf{e}_X \exp(ik_X x - i\omega t)\right\}$$

$$= \mathbf{e}_O \cos(k_O x - \omega t) + \mathbf{e}_X \cos(k_O x - \omega t + \Delta k x) \ , \ \Delta k \equiv k_X - k_O$$

  - the difference in wave number causes the O- and X-mode to drift in and out of phase with each other!

Modifying wave polarization in a quarter wave plate (3)

- The polarization when the wave exits the crystal at $x=L$

$$\mathbf{E}(t,x) = \left[\mathbf{e}_O \cos(k_O L - \omega t) + \mathbf{e}_X \cos(k_O L - \omega t + \Delta k L)\right] \ , \ \ \Delta k x = \pi/2$$

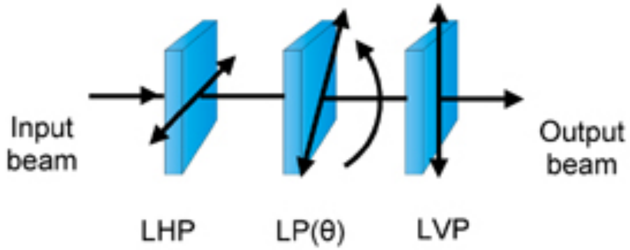
$$= \left[\mathbf{e}_O \cos(k_O L - \omega t) - \mathbf{e}_X \sin(k_O L - \omega t)\right]$$

$$L = \frac{c}{\omega}\frac{\pi/2}{\sqrt{K_\parallel} - \sqrt{K_\perp}}$$

  - This is cicular polarization!
  - The crystal converts linear to circular polarization (and vice versa)
  - Called a quarter wave plate; a common component in optical systems
  - But work *only* at one wave length – adapted for e.g. a specific laser!
  - In general, waves propagating in birefringent crystal change polarization back and forth between linear to circular polarization
  - Switchable wave plates can be made from liquid crystal
    - angle of polarization can be switched by electric control system
- Similar effect is Faraday effect in magnetoactive media
  - but the eigenmode are the circularly polarized components

Optical systems

- In optics, interferometry, polarometry, etc, there is an interest in following how the wave polarization changes when passing through e.g. an optical system.

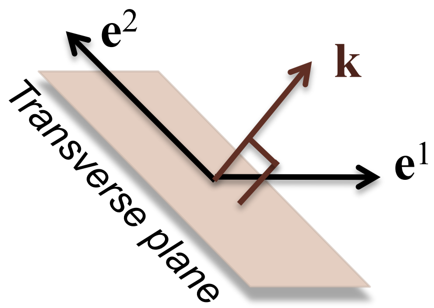


- For this purpose two types of calculus have been developed;
  - Jones calculus; only for coherent (polarized) wave
  - Muller calculus; for both coherent, unpolarised and partially polarised
- In both cases the wave is given by vectors $\mathbf{E}$ and $\mathbf{S}$ (defined later) and polarizing elements are given by matrixes J and M

$$\mathbf{E}_{out} = J \bullet \mathbf{E}_{in}$$
$$\mathbf{S}_{out} = M \bullet \mathbf{S}_{in}$$

The polarization of transverse waves

- Lets first introduce a new coordinate system representing vectors in the *transverse plane*, i.e. perpendicular to the $\mathbf{k}$.
  - Construct an orthonormal basis for $\{\mathbf{e}^1,\mathbf{e}^2,\kappa\}$, where $\kappa=\mathbf{k}/|\mathbf{k}|$
  - The transverse plane is then given by $\{\mathbf{e}^1,\mathbf{e}^2\}$, where



$$\mathbf{e}^\alpha = e_i^\alpha \mathbf{e}_i$$

  - where $\alpha=1,2$ and $\mathbf{e}_i$, $i=1,2,3$ is any basis for $\mathcal{R}^3$
  - denote $\mathbf{e}^1$ the *horizontal* and $\mathbf{e}^2$ the *vertical* directions
- The electric field then has different component representations: E_i (for $i=1,2,3$) and E^α (for $\alpha=1,2$)

$$E_i = e_i^\alpha E^\alpha$$

  - similar for the polarization vector, e_M

$$e_{M,i} = e_i^\alpha e_M^\alpha$$

The new coordinates provides 2D representations

Some simple Jones Matrixes

- In the new coordinate system the Jones matrix is 2x2:

$$J^{\alpha\beta} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$E^{\alpha}_{out} = J^{\alpha\beta} E^{\beta}_{in}$$

- Example: **Linear polarizer** transmitting horizontal polarization

$$J^{\alpha\beta}{}_{L,V} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_H \\ E_V \end{bmatrix} = \begin{bmatrix} E_H \\ 0 \end{bmatrix}$$

- Example: **Attenuator** transmitting a fraction ρ of the energy
  - Note: $energy \sim \varepsilon_0 |\mathbf{E}|^2$

$$J^{\alpha\beta}{}_{Att}(\rho) = \sqrt{\rho} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \sqrt{\rho} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E_H \\ E_V \end{bmatrix} = \sqrt{\rho} \begin{bmatrix} E_H \\ E_V \end{bmatrix}$$

Jones matrix for a quarter wave plate

- Quarter wave plates are birefringent (have two different refractive index)
  - align the plate such that horizontal / veritical polarization (corresponding to O/X-mode) has wave numbers $\mathbf{k}^1$ / $\mathbf{k}^2$

$$\begin{bmatrix} E_H(x) \\ E_V(x) \end{bmatrix} = \begin{bmatrix} E_H(0)\exp(ik^1 x) \\ E_V(0)\exp(ik^2 x) \end{bmatrix}$$

  - let the light entre the plate start at $x=0$ and exit at $x=L$

$$\mathbf{E}(L) = \begin{bmatrix} e^{ik^1 L} & 0 \\ 0 & e^{ik^2 L} \end{bmatrix} \begin{bmatrix} E_H(0) \\ E_V(0) \end{bmatrix} \equiv J_{Ph}\mathbf{E}(L)$$

  - where Ph stands for *phaser*
- Quarter wave plates chages the relative phase by $\pi/2$

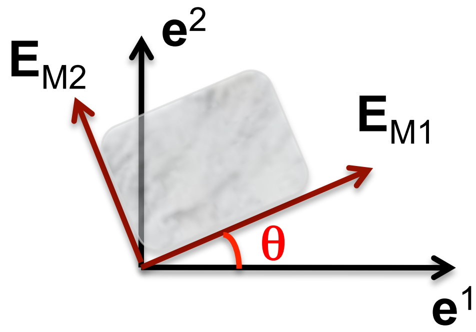
$$k^1 L - k^2 L = \pm\pi/2 \rightarrow J_Q = e^{ik^1 L} \begin{bmatrix} 1 & 0 \\ 0 & \pm i \end{bmatrix}$$

  - usually we considers only relative phase and skip factor $\exp(ik^1 L)$

Jones matrix for a rotated birefringent media

- If a birefringent media (e.g. quarter wave plates) is not aligned with the axis of our coordinate system with axis of the media
  - then we may use a rotation matrix

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \rightarrow R^{-1}(\theta) = R(-\theta)$$



  - Eigenmode have directions as in the fig.:

$$\mathbf{E}(x) = E_{M1}(x)\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} + E_{M2}(x)\begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

  - apply two rotation: first $-\theta$ and then $+\theta$, i.e. no net rotation:

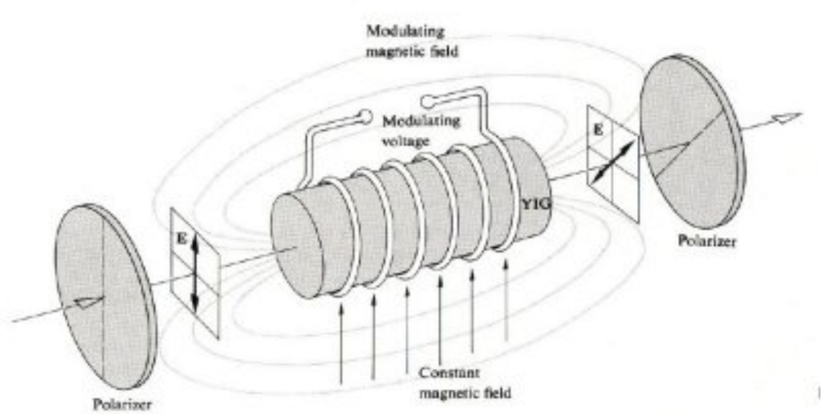
$$\mathbf{E}(x) = \underbrace{R(\theta)R(-\theta)}_{\text{no net rotation}}\left(E_{M1}(x)\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} + E_{M2}(x)\begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}\right) = R(\theta)\begin{bmatrix} E_{M1}(x) \\ E_{M2}(x) \end{bmatrix} =$$

$$= R(\theta)\begin{bmatrix} e^{ik^1x} & 0 \\ 0 & e^{ik^2x} \end{bmatrix}\begin{bmatrix} E_{M1}(0) \\ E_{M2}(0) \end{bmatrix} \rightarrow \boxed{J_{Ph} = R(\theta)\begin{bmatrix} e^{ik^1x} & 0 \\ 0 & e^{ik^2x} \end{bmatrix}}$$

Jones matrix for a Faraday rotation

- Faraday rotation is similar to birefrigency, except that eigenmodes have *circularly polarized eigenvector*

$$\mathbf{E}(x) = E_{M1}(x)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\end{bmatrix} + E_{M2}(x)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -i\end{bmatrix}$$



  - there is no useful rotation matrix
  - instead use a unitary matrix

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\ 1 & i\end{bmatrix} \rightarrow U^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ 1 & -i\end{bmatrix}$$

$$\mathbf{E}(x) = U^{-1}U\left(E_{M1}(x)\begin{bmatrix}1\\ i\end{bmatrix} + E_{M2}(x)\begin{bmatrix}1\\ -i\end{bmatrix}\right) = U^{-1}\begin{bmatrix}E_{M1}(x)\\ E_{M2}(x)\end{bmatrix} =$$

$$= U^{-1}\begin{bmatrix}e^{ik^1x} & 0\\ 0 & e^{ik^2x}\end{bmatrix}\begin{bmatrix}E_{M1}(0)\\ E_{M2}(0)\end{bmatrix} \rightarrow \boxed{J_{FR} = U^{-1}\begin{bmatrix}e^{ik^1x} & 0\\ 0 & e^{ik^2x}\end{bmatrix}}$$

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  - Stokes vector and polarization tensor
    - Poincare sphere
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Incoherent/unpolarised

- Many sources of electromagnetic radiation are not coherent
  - they do *not* radiate perfect harmonic oscillations (sinusoidal wave)
    - over short time scales the oscillations look harmonic
    - but over longer periods the wave look incoherent, or even stochastic
  - such waves are often referred to as unpolarised

- To model such waves we will consider the electric field to be a stochastic process, i.e. it has
  - an average: $< E^\alpha(t,\mathbf{x}) >$
  - a variance: $< E^\alpha(t,\mathbf{x})\, E^\beta(t,\mathbf{x}) >$
  - a covariance: $< E^\alpha(t,\mathbf{x})\, E^\beta(t+s,\mathbf{x}+\mathbf{y}) >$

- In this chapter we will focus on the variance, which we will refer to as the intensity tensor

$$I^{\alpha\beta} = < E^\alpha(t,\mathbf{x})\, E^\beta(t,\mathbf{x}) >$$

and the polarization tensor (where $\mathbf{e}_M = \mathbf{E} / |\mathbf{E}|$ is the polarization vector)

$$p^{\alpha\beta} = < e_M{}^\alpha(t,\mathbf{x})\, e_M{}^\beta(t,\mathbf{x}) >$$

The Stokes vector

- It can be shown that the intensity tensor is hermitian
  - thus it can be described by four Stokes parameter $\{I,Q,U,V\}$:

$$I^{\alpha\beta} = \frac{1}{2}\begin{bmatrix} I+Q & U-iV \\ U+iV & I-Q \end{bmatrix}$$

- A basis for hermitian matrixes is a set of unitary four matrixes:

$$\tau_1^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \tau_2^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \tau_3^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \tau_4^{\alpha\beta} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

  - where the last three matrixes are the Pauli matrixes
- Define the Stokes vector: $S_A=[I,Q,U,V]$

$$I^{\alpha\beta} = \frac{1}{2}\left[I\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + Q\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + U\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + V\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right]$$

$$I^{\alpha\beta} = \frac{1}{2}\tau_A^{\alpha\beta} S_A \quad \text{with inverse:} \quad S_A = \tau_A^{\alpha\beta} I^{\alpha\beta}$$

Representations for the polarization tensor

- The polarization tensor has similar representation
  - Note: $trace(p^{\alpha\beta})=1$, thus it is described by three parameter $\{q,u,v\}$:

$$p^{\alpha\beta} = \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + q\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + u\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + v\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right]$$

- As we will show in the following slides the four terms above represents different types of polarization
  - unpolarised (incoherent)
  - linear polarization
  - circular polarization

Examples

- For example consider:
  - linearly polarised wave $e_M{}^{\alpha}=[1,0]$

$$p^{\alpha\beta}=e_M^{\alpha}e_M^{\beta *}=\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1&0\end{bmatrix}=\begin{bmatrix}1&0\\0&0\end{bmatrix}=\frac{1}{2}\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}+1\begin{bmatrix}1&0\\0&-1\end{bmatrix}\right)$$

    - i.e. $\{q,u,v\}=\{1,0,0\}$
  - rotate linearly polarization by 45°, $e_M{}^{\alpha}=[1,1]\,/\,2^{1/2}$

$$p^{\alpha\beta}=e_M^{\alpha}e_M^{\beta *}=\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1&1\end{bmatrix}=\begin{bmatrix}1&1\\1&1\end{bmatrix}=\frac{1}{2}\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}+1\begin{bmatrix}0&1\\1&0\end{bmatrix}\right)$$

    - i.e. $\{q,u,v\}=\{0,1,0\}$
  - a circularly polarised wave, $e_M{}^{\alpha}=[1,-i]\,/\,2^{1/2}$

$$p^{\alpha\beta}=e_M^{\alpha}e_M^{\beta *}=\frac{1}{2}\begin{bmatrix}1\\i\end{bmatrix}\begin{bmatrix}1&-i\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1&-i\\i&1\end{bmatrix}=\frac{1}{2}\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}+1\begin{bmatrix}0&-i\\i&0\end{bmatrix}\right)$$

    - i.e. $\{q,u,v\}=\{0,0,1\}$

The polarization tensor for unpolarized waves (1)

- What are the Stokes parameters for unpolarised waves?
  - Let the e_M^1 and e_M^2 be independent stochastic variable

$$p^{\alpha\beta} = < \begin{pmatrix} e_M^1 \\ e_M^2 \end{pmatrix}^* \begin{pmatrix} e_M^1 & e_M^2 \end{pmatrix} > = \begin{bmatrix} < e_M^{1\,*} e_M^1 > & < e_M^{1\,*} e_M^2 > \\ < e_M^{2\,*} e_M^1 > & < e_M^{2\,*} e_M^2 > \end{bmatrix}$$

  - Since e_M^1 and e_M^2 are uncorrelated the offdiagonal term vanish

$$p^{\alpha\beta} = \begin{bmatrix} <| e_M^1 |^2> & 0 \\ 0 & <| e_M^2 |^2> \end{bmatrix}$$

  - The vector $\mathbf{e}_M$ is normalised: $\left|e_M^1\right|^2 + \left|e_M^2\right|^2 = 1$
  - By symmetry (no physical difference between e_M^1 and e_M^2)

$$\left|e_M^1\right|^2 = \left|e_M^2\right|^2 = 1/2$$

  - the polarization tensor then reads

$$p^{\alpha\beta} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

  - i.e. unpolarised have $\{q,u,v\}=\{0,0,0\}$!

The polarization tensor for unpolarized waves (2) ★

- Alternative derivation; polarization vector for unpolarized waves
  - Note first that the polarization vector is normalised

$$\left|\mathbf{e}_M\right|^2 = \left|e_M^1\right|^2 + \left|e_M^2\right|^2 = 1 \sim \cos^2(\theta) + \sin^2(\theta)$$

  - the polarization is complex and stochastic: $\begin{pmatrix} e_M^1 \\ e_M^2 \end{pmatrix} = \begin{pmatrix} e^{i\phi_1}\cos(\theta) \\ e^{i\phi_2}\sin(\theta) \end{pmatrix}$
    - where θ, ϕ_1 and ϕ_2 are uniformly distibuted in $[0,2\pi]$

- The corresponding polarization tensor

$$p^{\alpha\beta} =< \begin{pmatrix} e_M^1 \\ e_M^2 \end{pmatrix}^* \begin{pmatrix} e_M^1 & e_M^2 \end{pmatrix} >=< \begin{pmatrix} e^{-i\phi_1+i\phi_1}\cos(\theta)\cos(\theta) & e^{-i\phi_1+i\phi_2}\cos(\theta)\sin(\theta) \\ e^{-i\phi_2+i\phi_1}\sin(\theta)\cos(\theta) & e^{-i\phi_2+i\phi_2}\sin(\theta)\sin(\theta) \end{pmatrix} >$$

  - here the average is over the three random variables θ, ϕ_1 and ϕ_2

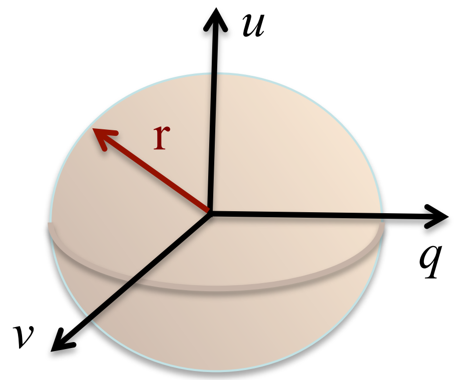
$$p^{\alpha\beta} = \frac{1}{(2\pi)^3}\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \begin{pmatrix} \cos^2(\theta) & e^{-i\phi_1+i\phi_2}\cos(\theta)\sin(\theta) \\ e^{-i\phi_2+i\phi_1}\sin(\theta)\cos(\theta) & \sin^2(\theta) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

  - i.e. unpolarised have $\{q,u,v\}=\{0,0,0\}$!

Poincare sphere

- The polarised part of a wave field describes the normalised vector $\{q/r, u/r, v/r\}$ where $r = \sqrt{q^2 + u^2 + v^2}$ is the *degree of polarization*
  - since this vector is real and normalised it will represent points on a sphere, the so called *Poincare sphere*

- Thus, any transverse wave field can be described by
  - a point on the Poincare sphere
  - a degree of polarization, r



Poincare sphere

- A polarizing element induces a motion on the sphere
  - e.g. when passing though a birefringent crystal we trace a circle on the Poincare sphere

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Weakly anisotropic media

- Next we will study Muller calculus for partially polorized waves
- We will do so for weakly anisotropic media:

$$K^{\alpha\beta} = n_0{}^2\delta^{\alpha\beta} + \Delta K^{\alpha\beta}$$

  - where $\Delta K^{\alpha\beta}$ is a small perturbation
  - although Muller calculus is *not* restricted to weak anisotropy
- The wave equation

$$\left(n^2 - n_0{}^2\right)E^{\alpha} = \Delta K^{\alpha\beta} E^{\alpha}$$

  - when ΔK_{ij} is a small, the 1st order dispersion relation reads: $n^2 \approx n_0{}^2$
  - the left hand side can then be expanded to give

$$n^2 - n_0^2 = (n-n_0)(n+n_0) = (n-n_0)n_0\left[2 + \underbrace{\frac{(n-n_0)}{n_0}}_{\text{small, <<1}}\right] \approx 2n_0(n-n_0)$$

$$2n_0(n-n_0)E^{\alpha} \approx \Delta K^{\alpha\beta} E^{\alpha}$$

The wave equation as an ODE

- Make an eikonal ansatz (assume $\mathbf{k}$ is in the x-direction):

$$E^\alpha = E_0^\alpha(t)\exp(ikx) = E_0^\alpha(t)\exp\left(i\frac{\omega}{c}n_0 x\right)\exp\left(i\frac{\omega}{c}(n-n_0)x\right)$$

$$\frac{dE^\alpha}{dx} = i\frac{\omega}{c}n_0 E^\alpha + i\frac{\omega}{c}\boxed{(n-n_0)E^\alpha}$$

same expression as on previous page!

- The wave equation can then be written as

$$\frac{dE^\alpha}{dx} = -i\frac{n_0\omega}{c}E^\alpha + i\frac{\omega}{2cn_0}\Delta K^{\alpha\beta}E^\alpha$$

  – describe how the wave changes when propagating through a media!

- Wave equation for the intensity tensor:

$$\frac{dI^{\alpha\beta}}{dx} = \frac{d}{dx} < E^\alpha E^{\beta^*} > = \ldots = \frac{i\omega}{2cn_0}\left(\Delta K^{\alpha\rho}\delta^{\beta\sigma} - \Delta K^{\beta\sigma^*}\delta^{\alpha\rho}\right)I^{\rho\sigma}$$

The wave equation as an ODE

- The wave equation the intensity tensor is not very convinient
- Instead, rewrite it in terms of the Stokes vector: $S_A = \tau_A^{\alpha\beta} I^{\alpha\beta}$

$$\frac{dS_A}{dx} = \left(\rho_{AB} - \mu_{AB}\right) S_B \qquad \begin{cases} \rho_{AB} = \dfrac{i\omega}{4cn_0}\left(\Delta K^{H,\alpha\rho}\tau_A^{\beta\alpha}\tau_B^{\rho\beta} - \Delta K^{H,\sigma\beta}\tau_A^{\rho\sigma}\tau_B^{\beta\rho}\right) \\ \mu_{AB} = \dfrac{i\omega}{4cn_0}\left(\Delta K^{H,\alpha\rho}\tau_A^{\beta\alpha}\tau_B^{\rho\beta} - \Delta K^{H,\sigma\beta}\tau_A^{\rho\sigma}\tau_B^{\beta\rho}\right) \end{cases}$$

  - we may call this the *differential formulation of Muller calculus*
  - symmetric matrix ρ_{AB} describes non-dissipative changes in polarization
  - and the antisymmetric matrix μ_{AB} describes dissipation (absorption)
- The ODE for S_A has the analytic solution (cmp to the ODE $y' = ky$)

$$S_A(x) = \left[\delta_{AB} + \left(\rho_{AB} - \mu_{AB}\right)x + 1/2\left(\rho_{AC} - \mu_{AC}\right)\left(\rho_{CB} - \mu_{CB}\right)x^2 + \ldots\right] S_B(0) =$$

$$= \exp\left[\left(\rho_{AB} - \mu_{AB}\right)x\right] S_B(0) = M_{AB} S_B(0)$$

  - where M_{AB} is called the *Muller matrix*
  - M_{AB} represents entire optical components
    - we have a *component based Muller calculus*

Examples of Muller matrixes

- For illustration only – don't memorise!

Linear polarizer
(Horizontal Transmission)

$$M_{AB}^{L,H} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Linear polarizer
(45° transmission)

$$M_{AB}^{L,45} = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Quarter wave plate
(fast axis horizontal)

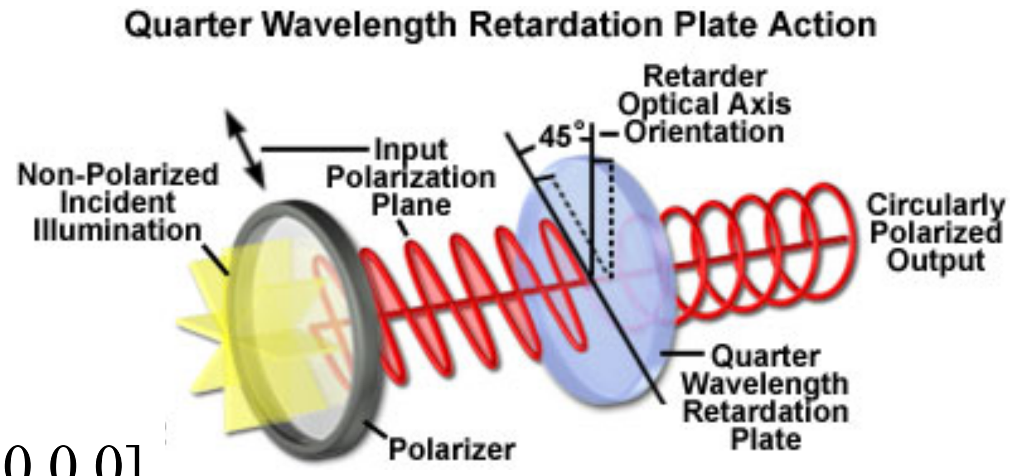
$$M_{AB}^{Q,H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Attenuating filter
(30% Transmission)

$$M_{AB}^{Att}(0.3) = 0.3\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Examples of Muller matrixes

- In optics it is common to connect a series of optical elements
- consider a system with:
  - a linear polarizer and
  - a quarter wave plate



$$S_A^{out} = M_{AB}^{Q,H} M_{BC}^{L,45} S_C^{in}$$

- Insert unpolarised light, $S_A^{in}=[1,0,0,0]$
  - **Step 1**: Linear polariser transmit linear polarised light

$$S^{step1} = M_{BC}^{L,45}\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^T = \begin{bmatrix}1 & 0 & -1 & 0\end{bmatrix}^T$$

  - **Step 2**: Quarter wave plate transmit circularly polarised light

$$S^{out} = M^{Q,H}\begin{bmatrix}1 & 0 & -1 & 0\end{bmatrix}^T = \begin{bmatrix}1 & 0 & 0 & -1\end{bmatrix}^T$$