11 Networked Control Systems

SOLUTION 11.1

Matrix Exponential The exponential of A, denote by e^A or $\exp(A)$, is the $n \times n$ matrix given by the power series

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \,.$$

The above series always converges, so the exponential of A is well defined. Note that if A is a 1×1 matrix, the matrix exponential of A is a 1×1 matrix consisting of the ordinary exponential of the signal element of A. Thus we have that

$$e^A = I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} ,$$

since $A * A = 0$.

We can find e^A via Laplace transform as well. As we know that the solution to the system linear differential equations given by

$$\frac{d}{dt} y(t) = Ay(t) \,, \quad y(0) = y_0 \,,$$

is

$$y(t) = e^{At} y_0 \,.$$

Using the Laplace transform, letting $Y(s) = \mathcal{L}\{y\}$, and applying to the differential equation we get

$$sY(s) - y_0 = AY(s) \Rightarrow (sI - A)Y(s) = y_0 \,,$$

where I is the identity matrix. Therefore,

$$y(t) = \mathcal{L}^{-1}\{(sI - A)^{-1}\} y_0 \,.$$

Thus, it can be concluded that

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\} \,,$$

from this we can find e^A by setting $t = 1$. Thus we can have

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\} = \mathcal{L}^{-1}\left\{ \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix} \right\} = \begin{bmatrix} u(t) & tu(t) \\ 0 & u(t) \end{bmatrix} .$$

We obtain the same result as before if we insert $t = 1$ into previous equation.

SOLUTION 11.2

Stability The eigenvalue equations for a matrix Φ is

$$\Phi v - \lambda v = 0 \,,$$

which is equivalent to

$$(\Phi - \lambda I)v = 0 \,,$$

where I is the $n \times n$ identity matrix. It is a fundamental result of linear algebra that an equation $Mv = 0$ has a non-zero solution v if and only if the determinant $\det(M)$ of the matrix M is zero. It follows that the eigenvalues of Φ are precisely the real numbers λ that satisfy the equation

$$\det(\Phi - \lambda I) = 0 \,.$$

The left-hand side of this equation can be seen to be a polynomial function of variable λ. The degree of this polynomial is n, the order of the matrix. Its coefficients depend on the entries of Φ, except that its term of degree n is always $(-1)^n\lambda^n$. For example, let Φ be the matrix

$$\Phi = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{bmatrix} .$$

The characteristic polynomial of Φ is

$$\det(\Phi - \lambda I) = \det\left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 4 \\ 0 & 0 & 9-\lambda \end{bmatrix} ,$$

which is

$$(2-\lambda)[(3-\lambda)(9-\lambda) - 16] = 22 - 35\lambda + 14\lambda^2 - \lambda^3 .$$

The roots of this polynomial are 2, 1, and 11. Indeed these are the only three eigenvalues of Φ, corresponding to the eigenvectors $[1,0,0]'$, $[0,2,-1]'$, and $[0,1,2]'$.

Given the matrix $\Phi = \text{diag}([-1.01, 1, -0.99])$, we plot following image, in which we can distinguish stable, asymptotical stable and instable state.

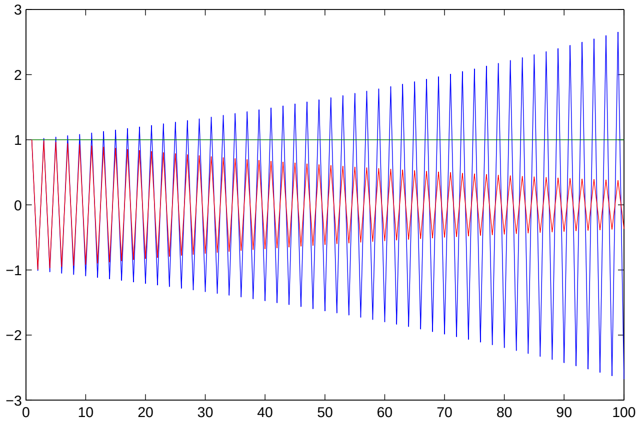

Figure 11.2.1: The stability, asymptotical stability and instability.

SOLUTION 11.3

Modeling

The dynamic for the state vector using Cartesian velocity, $(x, y, v_x, v_y, \omega)^T$, is given by:

$$\begin{aligned} \dot{x} &= v_x \\ \dot{y} &= v_y \\ \dot{v_x} &= -\omega v_y \\ \dot{v_y} &= \omega v_x \\ \dot{\omega} &= 0 . \end{aligned}$$