Emission from free particles

(Chapters 18-19)

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Outline

- Repetition: the emission formula and the multipole expansion
- The emission from a free particle - the Larmor formula
- Applications of the Larmor formula
  – Harmonic oscillator
  – Cyclotron radiation
  – Thompson scattering
  – Bremstrahlung
- Relativistic generalisation of Larmor formula
  – Repetition of basic relativity
  – Co- and contra-variant tensor notation
  – Lorentz transformation and relativistic invariants
  – Relativistic Larmor formula
- The Lienard-Wiechert potentials
  – Inductive and radiative electromagnetic fields
  – Alternative derivation of the Larmor formula
- Abraham-Lorentz force

Repetition: Emission formula

- The energy emitted by a wave mode M (using antihermitian part of the propagator), when integrating over the δ-function in ω

$$W_M = \sum_M \int d^3\mathbf{k}\, U_M(\mathbf{k})$$

$$U_M(\mathbf{k}) = \frac{R_M(\mathbf{k})}{\varepsilon_0} \left| \mathbf{e}_M^*(\mathbf{k}) \bullet \mathbf{J}_{ext}\left(\omega_M(\mathbf{k}),\mathbf{k}\right)\right|^2$$

  - the *emission formula*
  - thus U_M is a density of emission in $\mathbf{k}$-space
- Alternatively, the emission per unit frequency and solid angle ($d\Omega$)
  - for non-spatially dispersive media

$$W = \int \frac{d^2\Omega d\omega}{(2\pi)^3} \sum_M V_M(\omega,\Omega)$$

$$V_M(\omega,\Omega) = -\frac{\omega n_M}{(2\pi c)^3 \varepsilon_0} \frac{\left|\mathbf{e}_M^* \bullet \mathbf{J}_{\text{ext}}\right|^2}{1 - \left|\mathbf{e}_M^* \bullet \hat{\mathbf{k}}\right|^2}$$

Repetition: The current expressed in multipole moments

- Multipoles moments are related to Fourier transform of the current:

$$\mathbf{J}(\omega,\mathbf{k}) = \int dt e^{i\omega t} \int d^3\mathbf{x}\left[e^{-i\mathbf{k}\bullet\mathbf{x}}\right]\mathbf{J}(t,\mathbf{x})$$

$$= \int dt e^{i\omega t} \int d^3\mathbf{x}\left[1 - i\mathbf{k}\bullet\mathbf{x} - \frac{1}{2}(\mathbf{k}\bullet\mathbf{x})^2 + \ldots\right]\mathbf{J}(t,\mathbf{x})$$

$$= -i\omega\mathbf{d}(\omega) + i\mathbf{k}\times\mathbf{m}(\omega) - \omega\mathbf{q}\bullet\mathbf{k}(\omega)/2 + \ldots$$

Emission formula

(**k**-space power density)

$$\begin{cases} U_M^{\mathbf{d}}(\mathbf{k}) = \dfrac{R_M(\mathbf{k})}{\varepsilon_0}\left|\omega_M \mathbf{e}_M^* \bullet \mathbf{d}\right|^2 \\ U_M^{\mathbf{m}}(\mathbf{k}) = \dfrac{R_M(\mathbf{k})}{\varepsilon_0}\left|\mathbf{e}_M^* \bullet (\mathbf{k}\times\mathbf{m})\right|^2 \\ U_M^{\mathbf{q}}(\mathbf{k}) = \dfrac{R_M(\mathbf{k})}{4\varepsilon_0}\left|\omega_M \mathbf{e}_{M,i}^* q_{ij} k_j\right|^2 \end{cases}$$

Emission spectrum

(integrated over solid angles)

$$\begin{cases} V_{rad}^{\mathbf{d}}(\omega) = \dfrac{n(\omega)\omega^4}{6\pi^2\varepsilon_0 c^3}\left|\mathbf{d}(\omega)\right|^2 \\ V_{rad}^{\mathbf{m}}(\omega) = \dfrac{n(\omega)^3\omega^4}{6\pi^2\varepsilon_0 c^5}\left|\mathbf{m}(\omega)\right|^2 \\ V_{rad}^{\mathbf{q}}(\omega) = \dfrac{n(\omega)^3\omega^6}{720\pi^2\varepsilon_0 c^5} d_{ij} d_{ij}^* \end{cases}$$

Repetition: Emission in the time-domain

- The total radiated power from a dipole

$$P_{ave} = \frac{1}{6\pi\varepsilon_0 c^3}\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dt\left|\ddot{\mathbf{d}}(t)\right|^2\right) = \frac{1}{6\pi\varepsilon_0 c^3}\left\langle\left|\ddot{\mathbf{d}}(t)\right|^2\right\rangle$$

  - interpreted as a time average power
  - ideally the averaging should beperformed over all times
    - for event, or periodic motion, averaging can be done over finite times

Dipole current from single particle

- Current from a single particle: $\mathbf{J}(t,\mathbf{x}) = q\dot{\mathbf{X}}(t)\delta(x - \mathbf{X}(t))$

$$\mathbf{J}(\omega,\mathbf{k}) = q\int_{-\infty}^{\infty} dt e^{-i\omega t} \int d^3\mathbf{k}\, e^{i\mathbf{k}\bullet\mathbf{x}}\, \dot{\mathbf{X}}(t)\delta(\mathbf{x} - \mathbf{X}(t))$$

$$= -i\omega q\int_{-\infty}^{\infty} dt e^{-i\omega t} \underbrace{\left[1 + i\mathbf{k}\bullet\mathbf{X}(t) + \ldots\right]}\mathbf{X}(t)\delta(\mathbf{x} - \mathbf{X}(t))$$

$$= -i\omega q\left\{\mathbf{X}(\omega) + \int_{-\infty}^{\infty} dt e^{-i\omega t} \cancel{\left[i\mathbf{k}\bullet\mathbf{X}(t) + \ldots\right]}\mathbf{X}(t)\delta(\mathbf{x} - \mathbf{X}(t))\right\}$$

Dipole: $\mathbf{d}=q\mathbf{X}$

- Dipole approximation $\exp[ikx]\sim 1$; but what is the error?
  - Assume oscillating motion: $\dot{\mathbf{X}}(t) = \mathbf{v}\cos(\omega t) \Rightarrow \mathbf{X} \sim \mathbf{v}/\omega$
    - for non-oscillating motion: emission of quanta ω occures on time-scale $t\sim 1/\omega$

$$\mathbf{k}\bullet\mathbf{X}(t) \sim k\frac{v}{\omega} \sim \frac{n_M\omega}{c}\frac{v}{\omega} \sim n_M\frac{v}{c}$$

Dipole valid for non-relativistic motion

Relation between emission and acceleration

- Emission from single particle; dipole current: $\mathbf{J}(\omega,\mathbf{k}) \approx -i\omega q\mathbf{X}(\omega)$

$$V_M(\omega,\hat{\mathbf{k}}) = \frac{q^2 n_M}{\varepsilon_0 (2\pi c)^3} \frac{|\mathbf{e}_M^* \bullet \omega^2 \mathbf{X}(\omega)|^2}{1-|\mathbf{e}_M^* \bullet \hat{\mathbf{k}}|^2}$$

- Note that $-\omega^2\mathbf{X}(\omega) = \mathbf{a}(\omega)$ is the acceleration:

$$V_M(\omega,\hat{\mathbf{k}}) = \frac{q^2 n_M}{\varepsilon_0 (2\pi c)^3} \frac{|\mathbf{e}_M^* \bullet \mathbf{a}(\omega)|^2}{1-|\mathbf{e}_M^* \bullet \hat{\mathbf{k}}|^2}$$

  – Thus emission from free particle is a response to **acceleration**!

- Power radiated for isotropic transverse waves:

truncated outside [-T,T]

$$P_{rad}(\omega) = \lim_{T\to\infty} \frac{1}{2T} \int d\Omega \sum_M V_M^T(\omega,\hat{\mathbf{k}}) = \frac{q^2 n_M(\omega)}{12\pi^2\varepsilon_0 c^3} \lim_{T\to\infty} \frac{1}{2T} |\mathbf{a}^T(\omega)|^2$$

The Larmor fomula

- The emission integrated over all frequencies is related to time integral of the emitted power
  - in vacuume ($n_M=1$)

$$P \propto \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} \left|\mathbf{a}^T(\omega)\right|^2 d\omega = \lim_{T\to\infty} \frac{2\pi}{4T} \int_{-\infty}^{\infty} \left|\mathbf{a}^T(t)\right|^2 dt = \lim_{T\to\infty} \frac{2\pi}{T} \int_{-T}^{T} \left|\mathbf{a}(t)\right|^2 dt$$

The Larmor formula

$$P = \frac{q^2}{6\pi\varepsilon_0 c^3} \left\langle \left|\mathbf{a}(t)\right|^2 \right\rangle$$

  - the Larmor formula related the *averaged radiated power* with the *average acceleration*
- The time average has the same interpretation as for dipole and is sometimes written

$$P(t) = \frac{q^2}{6\pi\varepsilon_0 c^3} \left|\mathbf{a}(t)\right|^2$$

  - but should *always* be interpreted as an average!

Applications: Harmonic oscillator

- As a first example, consider the emission from a particle performing an harmonic oscillation
  - harmonic oscillations in one dimesion X

$$X(t) = x_0 \cos(\omega_0 t) \;,\; a(t) = \ddot{X}(t) = -x_0 \omega_0{}^2 \cos(\omega_0 t)$$

  - **Larmor formula**: the emitted power associated with this acceleration

$$P(t) = \frac{q^2 \omega_0{}^4 x_0{}^2}{6\pi\varepsilon_0 c^3} \cos^2\left(\omega_0 t\right)$$

  - oscillation $\cos^2(\omega_0 t)$ above is not physical; average over a period

$$\overline{P} = \frac{q^2 \omega_0{}^4 x_0{}^2}{12\pi\varepsilon_0 c^3}$$

Applications: Harmonic oscillator – frequency spectum

- Express the particle as a dipole $\mathbf{d}$, use truncation for Fourier transform

$$\mathbf{d}(\omega) = q\mathcal{F}\{\mathbf{X}(t)\} = \pi q\mathbf{x}_0\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right] =$$

$$= \lim_{T\to\infty} \frac{1}{2} Tq\mathbf{x}_0\left[\operatorname{sinc}\left(\frac{(\omega-\omega_0)T}{2}\right)+\operatorname{sinc}\left(\frac{(\omega+\omega_0)T}{2}\right)\right]$$

- The time-averaged power emitted from a dipole

$$P(\omega) = \lim_{T\to\infty}\frac{1}{T}\iint V(\omega,\Omega)\,d\Omega = \lim_{T\to\infty}\frac{1}{T}\frac{\left|\omega\mathbf{d}(\omega)\right|^2}{6\pi^2\varepsilon_0 c^3} =$$

$$= \lim_{T\to\infty}\frac{1}{T}\frac{1}{6\pi^2\varepsilon_0 c^3}\left|\frac{1}{2}\omega Tq\mathbf{x}_0\left[\operatorname{sinc}\left(\frac{(\omega-\omega_0)T}{2}\right)+\operatorname{sinc}\left(\frac{(\omega+\omega_0)T}{2}\right)\right]\right|^2 =$$

$$= \{\text{see Ch. } 4.5\}\ldots = \frac{q^2\omega_0^{\,4}x_0^{\,2}}{12\pi\varepsilon_0 c^3}\left[\delta\left(\omega-\omega_0\right)+\delta\left(\omega+\omega_0\right)\right]$$

Applications: cyclotron emission

- An important emission process from magnetised particles is from the acceleration involved in cyclotron motion
  - consider a charged particle moving in a static magnetic field $\mathbf{B}=B_z\mathbf{e}_z$

$$\mathbf{a}(t) = \dot{\mathbf{v}}(t) = \frac{q}{m}\mathbf{v}\times\mathbf{B} = -\Omega\mathbf{e}_z \times \mathbf{v} \quad , \quad \Omega = \frac{qB}{m}$$

$$\begin{bmatrix} \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{bmatrix} = \begin{bmatrix} 0 & \Omega & 0 \\ -\Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \Rightarrow \begin{bmatrix} v_x(t) \\ v_y(t) \\ v_x(t) \end{bmatrix} = \begin{bmatrix} v_\perp\sin(\Omega t) \\ v_\perp\cos(\Omega t) \\ v_\parallel \end{bmatrix} \Rightarrow \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \rho_L\cos(\Omega t) \\ \rho_L\sin(\Omega t) \\ v_\parallel t \end{bmatrix}$$

  - where we have the Larmor radius $\rho_L = v_\perp/\Omega$
  - This describe circular motion around magnetic field lines of force called the Larmor- or cyclotron-motion
  - The period averaged radiation: $\boxed{\overline{P} = \dfrac{q^4 B^2 v_\perp^2}{12\pi\varepsilon_0 c^3 m^2}}$
  - Magnetized plasma; power depends on the temperature: $\overline{P} \propto v_\perp^2 \propto T$
    - Electron cyclotron emission is one of the most common ways to measure the temperature of a fusion plasma!

Applications: wave scattering

- Consider a particle being accelerated by an external wave field

$$\mathbf{E}(t,\mathbf{x}) = \mathbf{E}_0 \cos(\omega_0 t - \mathbf{k}_0 \bullet \mathbf{x}) \;\Rightarrow\; \mathbf{a}(t) = q\mathbf{E}(t,\mathbf{x})/m$$

- The Larmor formula then tell us the average emitted power

$$\overline{P} = \frac{q^4 |\mathbf{E}_0|^2}{12\pi\varepsilon_0 m^2 c^3}$$

  – Note: that this is only valid in vacuum (restriction of Larmor formula)
- Rewrite in term of the wave energy density W_0
  – in vacuum : $W_0 = \varepsilon_0 |\mathbf{E}_0|^2/4 + \varepsilon_0 |\mathbf{k} \times \mathbf{E}_0 / c\omega|^2 / 4 = \varepsilon_0 |\mathbf{E}_0|^2 / 2$

$$\boxed{\overline{P} = \frac{8\pi}{3}\left(\frac{q^2}{4\pi\varepsilon_0 mc^2}\right)^2 cW_0}$$

  – Shown in the next page: This describe the fraction of the power density that is *scattered* by the particle, i.e. first absorbed and then re-emitted

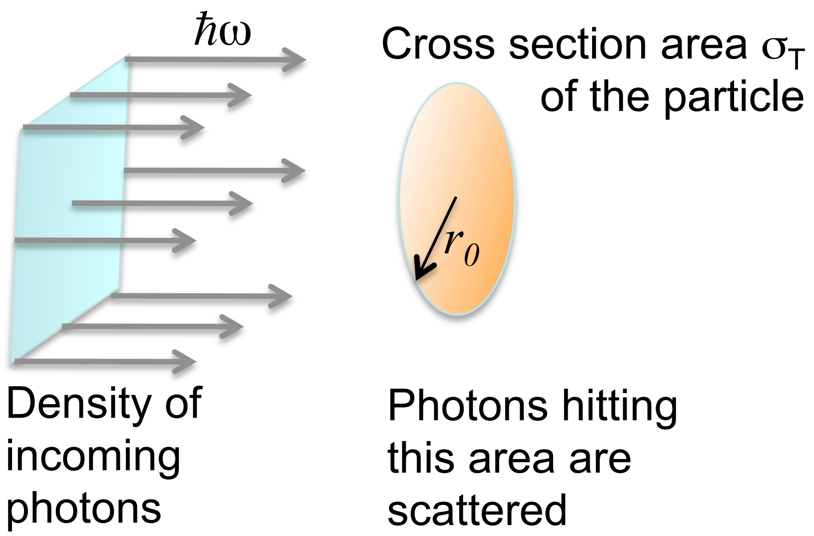
Applications: wave scattering

- The scattering process can be interpreted as a collision
  - Consider a density of wave quanta representing the power density W_0
  - The wave quanta, photons, move with velocity c (speed of light)
  - Imagine a charged particle as a ball with a cross section σ_T
  - Then the power scattered per unit time is given by

$$\overline{P} = \sigma_T c W_0$$

- The effective cross section for wave scattering on electrons

$$\sigma_T = \frac{8\pi}{3} r_0^2 \ , \quad r_0 = \frac{q^2}{4\pi\varepsilon_0 m c^2}$$



Applications: Thomson scattering

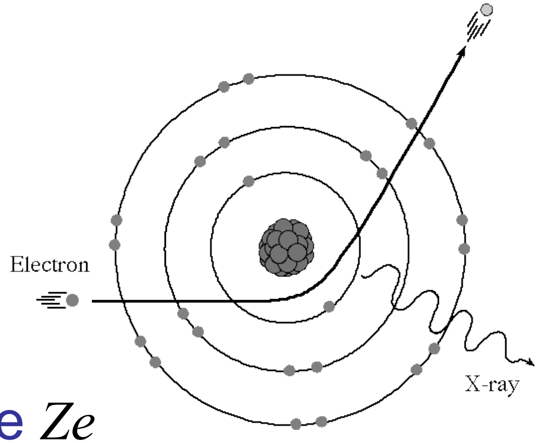
- Scattering of waves against *electrons* is called *Thomson scattering*
  - from this process the classical radius of the electron was defined as

$$r_{ele} = \frac{e^2}{4\pi\varepsilon_0 m_e c^2} \approx 2.8179 \times 10^{-15} [\mathrm{m}]$$

  - Note: this is an *effective radius* for Thomson scattering and not a measure of the “real” size of the electron
  - in general quantum mechanics is needed to understand the behavior of electrons at such short length scales
- Examples of Thomson scattering:
  - In fusion devices, Thomson scattering of a high-intensity laser beam is used for measuring the electron *temperatures* and *densities*.
  - The continous spectrum from the *solar corona* is the result of the Thomson scattering of solar radiation with free electrons
  - The *cosmic microwave background* is thought to be linearly polarized as a result of Thomson scattering

Applications: Bremsstrahlung

- *Bremsstrahlung* (~Braking radiation) come from the acceleration associated with electrostatic collisions between charged particles (called Coulomb collisions)
- Note that the electrostatic force is "long range", $E \sim 1/r^2$
  - thus electrostatic collisions between charged particles is a smooth continuous processes



- Derivation: an electron moving near an ion with charge Ze
  - since the ion is heavier than the electron, we assume $\mathbf{X}_{ion}(t)=0$
  - the equation of motion for the electron and the emitted power are

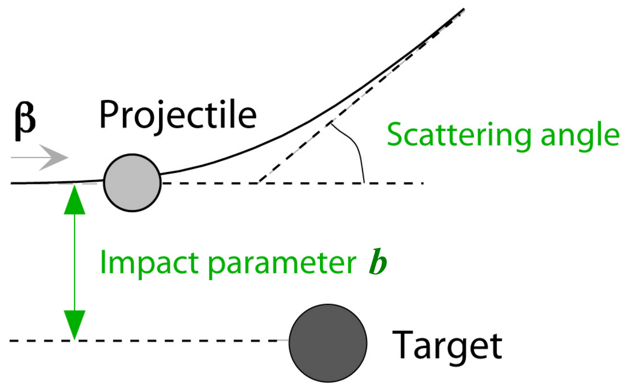
$$m_e \ddot{\mathbf{X}}(t) = -\frac{Ze^2 \mathbf{X}(t)}{4\pi\varepsilon_0 |\mathbf{X}(t)|^3} \quad \Rightarrow \quad P(t) = \frac{2Z^2}{3m_e^2 c^3} \left(\frac{e^2}{4\pi\varepsilon_0} \right)^3 \frac{1}{|\mathbf{X}(t)|^4}$$

  - this is the *Bremsstrahlung* radiation at one time of one single collision
    - to estimate the total power from a medium we need to integrate over both the entire collision and all ongoing collisions!

Bremsstrahlung: Coulomb collisions

- Lets try and integrate the emission over all times

$$W_{\mathrm{rad}} = \int_{-\infty}^{\infty} P(t)dt = 2\frac{2Z^2}{3m_e^2c^3}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^3 \int_{r_{\min}}^{\infty} \frac{dr}{r^4\dot{r}(t)}$$



  – where we integrate in the distance to the ion r
  – Now we need r_{min} and $\dot{r}(t)$
- So, let the ion be stationary at the origin
- Let the electron start at $(x,y,z)=(\infty,b,0)$ with velocity $\mathbf{v}=(-v_0,0,0)$
- The conservation of angular momentum and energy gives

$$\left.\begin{aligned} & m_e r^2\dot{\theta} = m_e b v_0 \\ & m_e\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{Ze^2}{4\pi\varepsilon_0 r} = \frac{1}{2}m_e {v_0}^2 \end{aligned}\right\} \Rightarrow \dot{r} = v_0\sqrt{1 + \frac{2b_0}{r} - \frac{b^2}{r^2}} \quad , \quad b_0 = \frac{Ze^2}{4\pi\varepsilon_0 m_e {v_0}^2}$$

  – This is the *Kepler problem* for the motion of the planets!
  – Next we need the minimum distance between ion and electron r_{min}

$$\dot{r}\,\Big|_{r=r_{\min}} = 0 \;\Rightarrow\; 1 + \frac{2b_0}{r_{\min}} - \frac{b^2}{{r_{\min}}^2} = 0 \;\Rightarrow\; {r_{\min}}^2 + 2b_0 r_{\min} - b^2 = 0 \;\Rightarrow\; r_{\min} = b_0 + \sqrt{{b_0}^2 + b^2}$$

Bremsstrahlung: Coulomb collisions

- Coulomb collisions are mainly due to "long range" interactions,
  - i.e. particles are far apart, and only slightly change their trajectories (there are exceptions in high density plasmas)
  - thus $r_{\min} \approx b$ and $b_0 << b$
  - we are then ready to evaluate the time integrated emission

$$W_{\text{rad}} = 2\frac{2Z^2}{3m_e^2c^3}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^3 \int_b^\infty \frac{dr}{r^4 v_0\sqrt{1+\frac{2b_0}{r}-\frac{b^2}{r^2}}} \approx \ldots \approx \frac{\pi Z^2}{3m_e^2c^3v_0b^3}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^3$$

- This is now the emission from a single collision
  - The cumulative emission from all particles and with all possible b and v_0 has no simple general solution (and is outside the scope of this course)
  - An approximate:

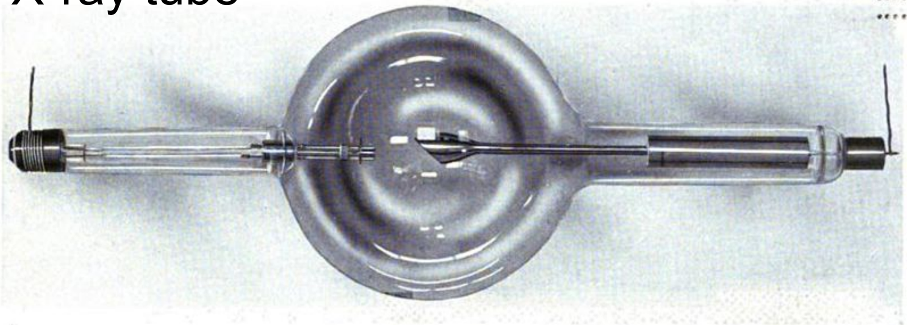
$$\boxed{P_{\text{Br}} \propto Z_i^2 n_i n_e \sqrt{T_e}}$$

    - Thus it can be used to derive information about both the charge, density and temperature of the media

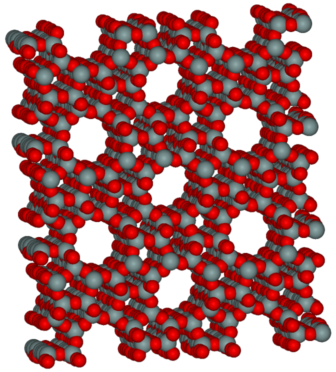
Industrial applications of Bremsstrahlung

- Typical frequency of Bremsstrahlung is in X-ray regime
- **X-ray tubes**: electrons are accelerated to high velocity
  When impacting on a metal surface they emit bremsstrahlung

X-ray tube



- X-ray tubes are also used in
  - CAT scanners
  - airport luggage scanners
  - X-ray crystallography
  - industrial inspection.



Applications of Bremsstrahlung

- **Astrophysics**: High temerature stellar objects T ~ 10^7-10^8 K radiate primarily in via bremsstrahlung
  - Note: surface of the sun 10^3 – 10^6 K

- **Fusion**:
  - Measurements of Bremsstrahlung provide information on the prescence of impurities with high charge, temperature and density
  - Bremsstrahlung and cyclotron radiation power losses:
    - Temperature at the centre of fusion plasma: ~10^8K ; the walls are ~10^3K
    - Main issue for fusion is to confine heat in plasma core
    - However, both Bremsstrahlung and cyclotron radiation escape easily
    - In reactor, radiation losses will be of importance – limits the reactor design
      - If plasma gets too hot, then radiation losses cool down the plasma.

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Quick recap of special relativity

- Next we'll derive a Larmor formula that is valid at relativistic velocities
- But first we'll recap the some basic theory of special relativity

- Special relativity is based on two postulates
  - **Principle of relativity**: The laws of physics are the same for all observers in uniform motion relative to one another
  - The **speed of light** in a vacuum (c = *299 792 458* m/s) is the same for all observers, regardless of their relative motion, or of the motion of the source of the light
- These postulates have many surprising consequences, e.g.
  - **Relativity of simultaneity**: Two events, simultaneous for one observer, may not be simultaneous for another observer if the observers are in relative motion.
  - **Time dilation**: Moving clocks are measured to tick more slowly than a "stationary" clock:

$$dT_{moving} = dT_{stationary}\sqrt{1 - v^2/c^2}$$

  - **Length contraction**: Objects are measured to be shortened in the direction that they are moving with respect to the observer.
  - **Mass–energy equivalence**: $E = mc^2$, energy and mass are equivalent and transmutable.

Background: Tensor formalism for special relativity

- The mathematical description of the "principle of relativity" is done with the so called *Lorentz transform*, which describe transformations between non-accelerated coordinate systems

- The Lorentz transform can be represented with the *Minkowski* formulation of *spacetime* [the 4D space spanned by time and real space (t,x,y,z)].

- In Minkowski spacetime every tensor has *two* representations
  - a vector F can be represented by co-variant components, F_μ
  - or by contra-variant components, F^μ
  - think of them as e.g. "row vector" & "column vector" components, the "bra" & "ket" of quantum mechanics, or as "dual" spaces

- The Minkowski spacetime is a vector space with the inner product

$$\langle F, G \rangle \equiv F^\mu G_\mu = F_\mu G^\mu = F_1 G^1 + F_2 G^2 + F_3 G^3 + F_4 G^4$$

  - i.e. here summation over repeated indexes is implicit
  - in this tensor formalism, index can be repeated *only* for multiplication of pairs of co- and contra-variant tensor components

Background: Tensor formalism for special relativity

- In the *Minkowski* formulation of spacetime the co- and contra-variant component of the position vector for the point (t, x, y, z)

$$x_\mu = [ct, -x, -y, -z]$$

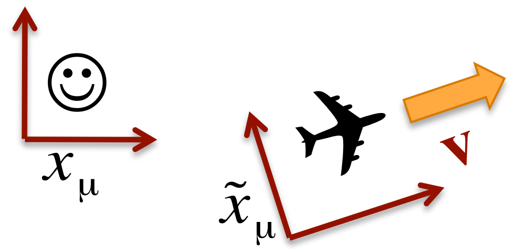
$$x^\mu = [ct, x, y, z]^T$$

  - thus $\|x\|^2 = x_\mu x^\mu = c^2 t^2 - |\mathbf{r}|^2$
  - Note: An alternative and equivalent definition is $x_\mu = [-ct, x, y, z]$
  - Strange? …we'll soon see why Minkowski choose this formulation
- Transformations between co- and contra-variant forms are performed with the metric tensor g^{mn} or g_{mn}

$$x_\mu = g_{\mu\nu} x^\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \Leftrightarrow x^\nu = g^{\nu\mu} x_\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Invariance of coordinate transformation

- Consider a moving object (moving in z-direction) viewed by an observer standing at the origin of a coordinate system x^μ & x_μ
  - The observer measures the speed of the object by measuring the position at two time points:
$$v = \left[z(T) - z(0)\right]/T$$
  - The length of the 4-vector of the plane from the first to the second measurement point is



$$\|dx\|^2 = dx^\mu dx_\mu = c^2 dt^2 - dz^2 = c^2T^2 - v^2T^2 = c^2T^2\left(1 - v^2/c^2\right)$$

  - Note: that $T\sqrt{1 - v/c}$ is the retarded time experienced by the object!
- Second coordinate system $\tilde{x}_\mu$ & $\tilde{x}^\mu$, where the "moving" object is in rest
  - measure the speed in the new coordinate system using the same time as the observer, i.e. $\|d\tilde{x}\|^2 = d\tilde{x}^\mu d\tilde{x}_\mu = c^2 d\tilde{t}^2 - 0$
  - equation for time-dilation: $d\tilde{t} = T\sqrt{1 - v^2/c^2} \Rightarrow \|d\tilde{x}\|^2 = c^2T^2\left(1 - v^2/c^2\right)$
- Thus, the length of a vector dx^μ is independent of the coordinate system, despite time dilation and length contraction!
  - We say that $dx^\mu dx_\mu$ is an *invariant* under *Lorentz* transformations

Lorentz transformations

- Principle of relativity reformulated:

The laws of physics are invariants under Lorentz transformations

- The transformation operator is therefore a 2-tensor $L^{\mu}{}_{\nu}$
  - a transformation equation from x^{μ} to $x^{\nu'}$ reads: $x^{\nu'} = L^{\nu'}{}_{\mu} x^{\mu}$
  - e.g. to a coordinate system moving in the x-direction with velocity $v=\beta c$:

$$L^{\nu'}{}_{\mu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{cases} t' = \gamma(t - \beta x) \\ x' = \gamma(x - \beta t) \\ y' = y \\ z' = z \end{cases}$$

Repeat calculation from previous slide:

$$dx' = 0 \Rightarrow dx = \beta dt$$
$$\Rightarrow dt' = \gamma(dt - \beta\beta dt)$$
$$\ldots = dt/\gamma$$

- Invariance of inner product: $a^{\mu'} b_{\mu'} = a^{\lambda} b_{\lambda}$

$$\left.\begin{aligned} a^{\mu'} b_{\mu'} &= g_{\mu'\nu'} a^{\mu'} b^{\nu'} = g_{\mu'\nu'} \left(L^{\mu'}{}_{\lambda} a^{\lambda}\right)\left(L^{\nu'}{}_{\eta} b^{\eta}\right) \\ a^{\lambda} b_{\lambda} &= g_{\eta\lambda} a^{\lambda} b^{\eta} \end{aligned}\right\}$$

L has to satisfy:

$$\boxed{g_{\mu'\nu'} L^{\mu'}{}_{\eta} L^{\nu'}{}_{\lambda} = g_{\eta\lambda}}$$

Representation of physical quantities in special relativity

- For formulating physical laws we need 4-vector generalisation of common physical quantities; careful considerations yields
  - position $x^\mu = [ct, \mathbf{x}]$
  - velocity $u^\mu = [\gamma c, \gamma \mathbf{v}]$
  - momentum $p^\mu = [\varepsilon / c, \mathbf{p}]$, $\varepsilon^2 = m^2 c^4 + |\mathbf{p}|^2 c^2$
  - wave vector $k^\mu = [\omega / c, \mathbf{k}]$
  - current density $J^\mu = [\rho c, \mathbf{J}]$
  - 4-vector potential $A^\mu = [\phi / c, \mathbf{A}]$
  - Force $F^\mu = [\gamma \mathbf{v} \cdot \mathbf{F} / c, \gamma \mathbf{F}]$

  where $\gamma = \left(1 - v^2 / c^2\right)^{-1/2}$
- All these quantities has co-variant representations, e.g. $A_\mu = g_{\mu\nu} A^\nu$
- Any inner product between two of these quantities are invariant under Lorentz transformations!

Relativistic Larmor formula

- In special relativity the velocity and acceleration is defined in the frame of an arbitrary observer experiencing a time τ

$$v^{\mu}(\tau) = \frac{\partial x^{\mu}(\tau)}{\partial \tau} \qquad a^{\mu}(\tau) = \frac{\partial^2 x^{\mu}(\tau)}{\partial \tau^2}$$

$$\Rightarrow$$

$$v^{\mu}(\tau) = [c, \dot{x}, \dot{y}, \dot{z}]^T$$
$$v_{\mu}(\tau) = [c, -\dot{x}, -\dot{y}, -\dot{z}]$$
$$a^{\mu}(\tau) = [0, \ddot{x}, \ddot{y}, \ddot{z}]^T$$
$$a_{\mu}(\tau) = [0, -\ddot{x}, -\ddot{y}, -\ddot{z}]$$

  - Since inner products are invariant under Lorentz transformations, thus $a^{\mu}(\tau)a_{\mu}(\tau) = -\ddot{x}^2 - \ddot{y}^2 - \ddot{z}^2 = -|\mathbf{a}|^2$ is an invariant!
- Consider a single particle, then pick a system where it is at rest at time τ, but being accelerated by a force
  - In this coordinate system the particle is non-relativistic and the (average) emitted power is given by the Larmor formula

$$\boxed{P(\tau) = -\frac{q^2}{6\pi\varepsilon_0 c^3} a^{\mu}(\tau)a_{\mu}(\tau)}$$

  - since $a^{\mu}a_{\mu}$ is an invariant, so is $P(\tau)$, i.e. above we have a *relativistic Larmor formula*!!

Relativistic Larmor formula

- Rewrite the acceleration in term of the 3-vector velocity $\mathbf{v}$

$$a^{\mu} = \gamma^2 \left[\frac{\gamma^2}{c} \mathbf{v} \bullet \dot{\mathbf{v}} \quad , \quad \dot{\mathbf{v}} + \frac{\gamma^2}{c^2} \mathbf{v}(\mathbf{v} \bullet \dot{\mathbf{v}}) \right]$$

$$a^{\mu} a_{\mu} = -\gamma^2 \left(|\dot{\mathbf{v}}|^2 - \frac{|\mathbf{v} \times \dot{\mathbf{v}}|^2}{c^2} \right)$$

$$\Rightarrow P = \frac{q^2}{6\pi\varepsilon_0 c^3} \gamma^2 \left(|\dot{\mathbf{v}}|^2 - \frac{|\mathbf{v} \times \dot{\mathbf{v}}|^2}{c^2} \right)$$

- Acceleration in special relativity is somewhat artificial
  - instead use the force
  - Larmor formula in terms of the 3-vector force $\mathbf{F}$

$$P = \frac{q^2}{6\pi\varepsilon_0 c} \frac{\gamma^2}{m^2 c^2} \left(|\mathbf{F}|^2 - \frac{|\mathbf{v} \times \mathbf{F}|^2}{c^2} \right)$$

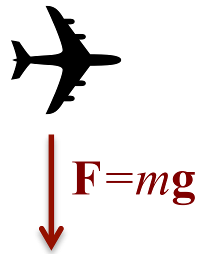
Note: there's a typo in the book; the cross product is replaced by scalar product

Relativistic Larmor formula

- Let the object move perpendicular to the force

$$|\mathbf{v}\times\mathbf{F}| = |\mathbf{v}||\mathbf{F}|$$

$$P = \gamma^2 \frac{q^2}{6\pi\varepsilon_0 c^3 m^2}\mathbf{F}^2 \underbrace{\left(1-\frac{|\mathbf{v}|^2}{c^2}\right)}_{\gamma^{-2}} = \frac{q^2}{6\pi\varepsilon_0 c^3 m^2}\mathbf{F}^2$$

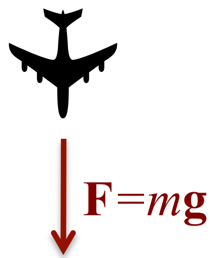


  – No relativistic correction!

- Let the object move along the force
  – i.e. $\mathbf{F}$ is parallel to $\mathbf{v}$

$$\mathbf{v}\times\mathbf{F} = 0$$

$$P = \gamma^2 \frac{q^2}{6\pi\varepsilon_0 c^3 m^2}\mathbf{F}^2$$



  – relativitstic correction by γ^2 !

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  - Bremstrahlung
- Relativistic generalisation of Larmor formula
  - Repetition of basic relativity
  - Co- and contra-variant tensor notation
  - Lorentz transformation and relativistic invariants
  - Relativistic Larmor formula
- The Lienard-Wiechert potentials
  - Inductive and radiative electromagnetic fields
  - Alternative derivation of the Larmor formula
- Abraham-Lorentz force

Chapter 19: Alternative treatments of emission processes

- The traditional treatment of emission; study the emitted Poynting flux
  - start with the scalar and vector potentials from single particle
- Lorentz gauge: the potentials follows d'Alemberts equation (Ch. 5)

$$\begin{bmatrix} \mathbf{A}(t,\mathbf{x})/\mu_0 \\ \phi(t,\mathbf{x})\varepsilon_0 \end{bmatrix} = \int dt'\, d^3\mathbf{x}' \frac{\delta\left(t - t' - |\mathbf{x} - \mathbf{x}'|/c\right)}{4\pi|\mathbf{x} - \mathbf{x}'|} \begin{bmatrix} \mathbf{J}(t',\mathbf{x}') \\ \rho(t',\mathbf{x}') \end{bmatrix}$$

- When the sources is from a single particle

$$\begin{bmatrix} \mathbf{A}(t,\mathbf{x})/\mu_0 \\ \phi(t,\mathbf{x})\varepsilon_0 \end{bmatrix} = \int dt' d^3\mathbf{x}' \frac{\delta\left(t - t' - |\mathbf{x} - \mathbf{x}'|/c\right)}{4\pi|\mathbf{x} - \mathbf{x}'|} \begin{bmatrix} q\dot{\mathbf{X}}(t')\delta\left(\mathbf{x}' - \mathbf{X}(t')\right) \\ q\delta\left(\mathbf{x}' - \mathbf{X}(t')\right) \end{bmatrix}$$

  - integrate over $\mathbf{x}'$

$$\begin{bmatrix} \mathbf{A}(t,\mathbf{x})/\mu_0 \\ \phi(t,\mathbf{x})\varepsilon_0 \end{bmatrix} = \int dt' \frac{\delta\left(t - t' - |\mathbf{x} - \mathbf{X}(t')|/c\right)}{4\pi|\mathbf{x} - \mathbf{X}(t')|} \begin{bmatrix} q\dot{\mathbf{X}}(t') \\ q \end{bmatrix}$$

The Lienard-Wiechert potentials

- Be careful with integration over t' !
  - Note that t' appear non-linearly in the Dirac delta
    - Remember: $$\int dx\,\delta\big(g(x)\big) = \sum_{x_r : g(x_r)=0} \big(g'(x_r)\big)^{-1}$$
    - thus $$\int dt'\,\delta\big(t - t' - |\mathbf{x} - \mathbf{X}(t')|/c\big) = \left[\frac{\partial}{\partial t_r}\big(t - t_r - |\mathbf{x} - \mathbf{X}(t_r)|/c\big)\right]^{-1}$$
      $$= \ldots = \left[1 - \frac{\big(\mathbf{x} - \mathbf{X}(t_r)\big)\bullet\dot{\mathbf{X}}(t_r)}{c|\mathbf{x} - \mathbf{X}(t_r)|^2}\right]^{-1}$$, where $t - t_r - |\mathbf{x} - \mathbf{X}(t_r)|/c = 0$
  - the time t_r is known as the *retarded time*
    - Note: $t - t_r - |\mathbf{x} - \mathbf{X}(t_r)|/c = 0$ is a non-linear equation for t_r
- The potentials can then be written as the *Lienard-Wiechert* potentials

$$\begin{bmatrix} \mathbf{A}(t,\mathbf{x})/\mu_0 \\ \phi(t,\mathbf{x})\varepsilon_0 \end{bmatrix} = \frac{q}{4\pi\left\| \mathbf{x} - \mathbf{X}(t_r)\right| - \big(\mathbf{x} - \mathbf{X}(t_r)\big)\bullet\dot{\mathbf{X}}(t_r)/c\right|} \begin{bmatrix} \dot{\mathbf{X}}(t') \\ 1 \end{bmatrix}$$

The Lienard-Wiechert potentials

- The *Lienard-Wiechert* potentials are simplified when choosing coordinate to locate the source to the origin $\mathbf{X}(t_r) = 0$

$$\begin{bmatrix} \mathbf{A}(t,\mathbf{x})/\mu_0 \\ \phi(t,\mathbf{x})\varepsilon_0 \end{bmatrix} = \frac{q}{4\pi\left(|\mathbf{x}| - \mathbf{x} \bullet \dot{\mathbf{X}}(t_r)/c\right)} \begin{bmatrix} \dot{\mathbf{X}}(t_r) \\ 1 \end{bmatrix}$$

  - Note that the term $\mathbf{x} \bullet \dot{\mathbf{X}}(t_r)/c \sim v/c$ is a relativistic term
  - The Lienard-Wiechert potentials are derived from Maxwells equations, thus they are automatically relativistically correct!

E and B from the Lienard-Wiechert potentials

- Calculate E and B field from the *Lienard-Wiechert* potentials

$$\mathbf{E} = -\nabla\phi + \frac{\partial \mathbf{A}}{\partial t} \quad , \quad \mathbf{B} = \nabla \times \mathbf{A}$$

  - Note functional dependence $t_r = t_r(t,\mathbf{x})$ thus

$$\nabla\phi(t_r(t,\mathbf{x}),\mathbf{x}) = \frac{\partial\phi(t_r,\mathbf{x})}{\partial t_r}\nabla t_r + \mathbf{e}_j \frac{\partial}{\partial x_j}\phi(t_r,\mathbf{x})$$

$$\nabla \bullet \mathbf{A}(t_r(t,\mathbf{x}),\mathbf{x}) = \frac{\partial \mathbf{A}(t_r,\mathbf{x})}{\partial t_r} \bullet \nabla t_r + \frac{\partial}{\partial x_j} A_j(t_r,\mathbf{x})$$

$$\frac{\partial}{\partial t} A_i(t_r(t,\mathbf{x}),\mathbf{x}) = \frac{\partial A_i(t_r,\mathbf{x})}{\partial t_r}\frac{\partial t_r}{\partial t}$$

– After lengthy calculations, using $\{r \equiv |\mathbf{x}| \ , \ \mathbf{n} \equiv \mathbf{x}/|\mathbf{x}| \ , \ \mathbf{a} \equiv \mathbf{a}(t_r) \ , \ \mathbf{v} \equiv \mathbf{v}(t_r)\}$

$$\mathbf{B} = \frac{\mathbf{n}\times\mathbf{E}}{c} \quad , \quad \mathbf{E} = \frac{q}{4\pi\varepsilon_0}\frac{(\mathbf{x} - r\mathbf{v}/c)\left(1 - v^2/c^2 + \mathbf{x}\bullet\mathbf{a}/c^2\right) - \left(\mathbf{a}r/c^2\right)(r - \mathbf{x}\bullet\mathbf{v}/c)}{(r - \mathbf{x}\bullet\mathbf{v}/c)^3}$$

Radiative and Inductive fields

- The electric field has term $\sim 1/r$ and other $\sim 1/r^2$; use $\mathbf{x}=\mathbf{n}r$

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0} \frac{\left[(\mathbf{n}-\mathbf{v}/c)\left(1-v^2/c^2+r\mathbf{n}\bullet\mathbf{a}/c^2\right)-r\left(\mathbf{a}/c^2\right)\left(\left(1-\mathbf{n}\bullet\mathbf{v}/c\right)\right)\right]}{r^2\left(1-\mathbf{n}\bullet\mathbf{v}/c\right)^3} =$$

$$= \frac{q}{4\pi\varepsilon_0\left(1-\mathbf{n}\bullet\mathbf{v}/c\right)^3}\left\{\underbrace{\frac{(\mathbf{n}-\mathbf{v}/c)\mathbf{n}\bullet\mathbf{a}-\left(1-\mathbf{n}\bullet\mathbf{v}/c\right)\mathbf{a}}{c^2 r}}_{\text{Radiative}} + \underbrace{\frac{(\mathbf{n}-\mathbf{v}/c)\left(1-v^2/c^2\right)}{r^2}}_{\text{Inductive}}\right\}$$

  - Note: only the radiative terms depend on the acceleration
- Imagine radiation as emitted photons;
  - Radiate N photons at $t=0$, $|\mathbf{x}|=0$, then at time $T>0$ you have N photons on the sphere with radius $R=cT$, i.e. the number of photons is independent of R.
- What energy (~number of photons) reach the sphere of radius R?
  - Radiative $W \sim \oint |\mathbf{E}|^2 R^2 d\Omega \sim |1/R|^2 R^2 \sim 1$ for all R - free moving photons
  - Inductive $W \sim \oint |\mathbf{E}|^2 R^2 d\Omega \sim \left|1/R^2\right|^2 R^2 \sim R^{-2}$ *virtual photons* bound to not leave the particle unless absorbed by receiver particle

Simplified form for the electric field

- Simplify the *radiative* electric field (for improved interpretation)

$$\mathbf{E}_{\text{rad}} = \frac{q}{4\pi\varepsilon_0(1-\mathbf{n}\bullet\mathbf{v}/c)^3}\left\{\frac{(\mathbf{n}-\mathbf{v}/c)\mathbf{n}\bullet\mathbf{a}-(1-\mathbf{n}\bullet\mathbf{v}/c)\mathbf{a}}{c^2 r}\right\}$$

- Simplify the numerator: $\mathbf{e}_j \bullet \{(\mathbf{n}-\mathbf{v}/c)\mathbf{n}\bullet\mathbf{a}-(1-\mathbf{n}\bullet\mathbf{v}/c)\mathbf{a}\} =$

$$= \{n_k n_j - n_k v_j/c - (1-n_m v_m/c)\delta_{jk}\}a_k =$$

$$= [n_k n_j - \delta_{jk}]a_k - [n_k v_j - n_m v_m \delta_{jk}]a_k/c$$

- Use the vector identity:

$$[\mathbf{a}\times(\mathbf{b}\times\mathbf{c})]_i = \varepsilon_{ijk}\varepsilon_{klm}a_j b_l c_m = (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})a_j b_l c_m = (a_j b_i - a_j b_j)c_j$$

$$\Rightarrow [n_k n_j - \delta_{jk}]a_k = [\mathbf{n}\times(\mathbf{n}\times\mathbf{a})]_j \quad \& \quad [n_k v_j - n_m v_m\delta_{jk}]a_k = [\mathbf{n}\times(\mathbf{v}\times\mathbf{a})]_j$$

$$\boxed{\mathbf{E}_{\text{rad}} = \frac{q}{4\pi\varepsilon_0 c^2}\frac{\mathbf{n}\times[(\mathbf{n}-\mathbf{v}/c)\times\mathbf{a}]}{(1-\mathbf{n}\bullet\mathbf{v}/c)^3 r}}$$

$$\boxed{\mathbf{B}_{\text{rad}} = \frac{q}{4\pi\varepsilon_0 c^3}\frac{\mathbf{n}\times\{\mathbf{n}\times[(\mathbf{n}-\mathbf{v}/c)\times\mathbf{a}]\}}{(1-\mathbf{n}\bullet\mathbf{v}/c)^3 r}}$$

The Poynting flux

- Poynting flux radiated by a single charge particle, which at the retarded time t_r had velocity $\mathbf{v}$ and acceleration $\mathbf{a}$

$$\mathbf{F}_{\mathrm{EM}} = \frac{\mathbf{E}\times\mathbf{B}}{\mu_0} = \ldots = \frac{q^2}{16\pi^2\varepsilon_0 c^2}\frac{\left|\mathbf{n}\times\left[(\mathbf{n}-\mathbf{v}/c)\times\mathbf{a}\right]\right|^2}{\left(1-\mathbf{n}\bullet\mathbf{v}/c\right)^6 r^2}\mathbf{n}$$

- The non-relativistic Poynting flux:

$$\mathbf{F}_{\mathrm{EM}} = \frac{q^2}{16\pi^2\varepsilon_0 c^2}\frac{\left|\mathbf{n}\times\left[\mathbf{n}\times\mathbf{a}\right]\right|^2}{r^2}\mathbf{n} + O(v/c)$$

Radiated power – non-relativistic limit

- The Poynting flux is an energy flux
  - The flux through a closed surface = the power leaving the enclosed region
- Encircle the particle with a large sphere with radius R
  - The power leaving this sphere is

$$P = \oiint_{r=R} d\mathbf{S} \bullet \mathbf{F}_{\mathrm{EM}} = \frac{q^2}{16\pi^2 \varepsilon_0 c^2} \oiint d\Omega \left| \mathbf{n} \times [\mathbf{n} \times \mathbf{a}] \right|^2 + O(v/c)$$

$$\left\{ \begin{aligned} &\oiint d\Omega \left| \mathbf{n} \times [\mathbf{n} \times \mathbf{a}] \right|^2 = \oiint d\Omega \left[\left(n_k n_j - \delta_{jk} \right) a_k \right] \left[\left(n_m n_j - \delta_{jm} \right) a_m \right] = \oiint d\Omega \left[a^2 - \left(n_k a_k \right)^2 \right] = \\ &= \left\{ \mathrm{let} : n_k a_k = a\cos(v) \right\} = a^2 \oiint d\Omega \sin^2(v) = \left\{ x = \sin(v) \right\} = \int_0^{2\pi} d\phi \int_{-1}^{1} dx x^2 = \frac{4}{3}\pi a^2 \end{aligned} \right\}$$

- The non-relativistic power leaving the sphere is given by

$$P(t,R) = \frac{q^2 a(t_r)^2}{6\pi\varepsilon_0 c^2} + O(v/c)$$

the radiated power at the retarded time t_r as given by the Larmor formula!

Sketch of emission in dispersive media

- The Poynting flux approach to emission can also be used in dispersive media, however this procedure may become tedious
- Start with the photon propagator for the wave equation

$$A_i(t,\mathbf{x}) = \int d\omega d^3\mathbf{k} \frac{\mu_0 c^2}{\omega^2} \frac{\lambda_{ij}}{\Lambda} J_j(\omega,\mathbf{k}) e^{-i\omega t + i\mathbf{k}\bullet\mathbf{x}}$$

- Particle source $J_i(\omega,\mathbf{k}) = q\omega^2 X_i(\omega,\mathbf{k}) = q\omega^2 \int dt e^{i\omega t - i\mathbf{k}\bullet\mathbf{X}(t)} X_i(t)$

$$A_i(t,\mathbf{x}) = q\mu_0 c^2 \int dt' X_i(t') \int d\omega d^3\mathbf{k} \frac{\lambda_{ij}}{\Lambda} e^{-i\omega(t-t') + i\mathbf{k}\bullet(\mathbf{x}-\mathbf{X}(t'))}$$

- The emitted power is now given by the energy flux, which now include both an electro-magnetic term and a particle term (Ch. 15)

$$P_{flux} = \oiint_{r=R} d\Omega \left(\mathbf{F}_M^{EM} + \mathbf{F}_M^P\right) = \oiint_{r=R} d\Omega \frac{\omega_M(\mathbf{k})}{\mu_0} \left\{ 2\Re\left[A^2\mathbf{k} - \mathbf{A}\mathbf{A}^* \bullet \mathbf{k}\right] - A_i^* \frac{\partial K_{ij}}{\partial \mathbf{k}} A_j \right\}$$

- However, this formalism is often more tedious then treating emission in terms of the "work" as outlined earlier in this lecture.

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Self-forces

- **Energy conservation**: A particle emitting radiation losses energy

$$E_2 - E_1 = \hbar\omega$$ (*from previous lecture – from quantum mechanics*)

- Interpretation: the emitted wave performs work on the particle:

$$W = \mathbf{F} \bullet \mathbf{v} = q\mathbf{E} \bullet \mathbf{v} = \mathbf{E} \bullet \mathbf{J}$$

  - The force from one particle onto itself is called the *self-force*
- Remember your first lecture on electrostatics
  - two charged particles “1” and “2” exert a force on each other

$$\mathbf{F}_1 = -\mathbf{F}_2 = \frac{q_1 q_2 (\mathbf{r}_2 - \mathbf{r}_1)}{4\pi\varepsilon_0 |\mathbf{r}_2 - \mathbf{r}_1|^3}$$

- This says nothing about the force from a single particle on itself, i.e. the self-force, that is needed to describe emission
  - …this problem is only properly resolved in quantum mechanics!
- Here we’ll derive the *Abraham-Lorentz force*
  - classical treatment of radiation conserving energy and momentum

Radiation reaction

- This power emitted during radiation, according to the Larmor formula

$$P(t) = \frac{q^2 |\dot{\mathbf{v}}(t)|^2}{6\pi\varepsilon_0 c^3}$$

- Construct a *radiation-reaction force* that describe the lost energy
  - the work by this force: $P(t) = -\mathbf{v}(t) \bullet \mathbf{F}_{react}(t)$
- Time integration

$$\int_{t_1}^{t_2} P(t)dt = \frac{q^2}{6\pi\varepsilon_0 c^3} \int_{t_1}^{t_2} \dot{v}(t)^2 dt = -\frac{q^2}{6\pi\varepsilon_0 c^3} \left\{ \left[\mathbf{v}(t) \bullet \dot{\mathbf{v}}(t) \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} dt \mathbf{v}(t) \bullet \ddot{\mathbf{v}}(t) \right\}$$

- To remove the first term, consider e.g. periodic motion, or events such that the acceleration is finite during event

$$\int_{t_1}^{t_2} P(t)dt = -\frac{q^2}{6\pi\varepsilon_0 c^3} \int_{t_1}^{t_2} dt \mathbf{v}(t) \bullet \ddot{\mathbf{v}}(t)$$

Abraham-Lorentz equation of motion

- Let the integrated power be represented by a reaction force

$$\int_{t_1}^{t_2} P(t)dt = -\frac{q^2}{6\pi\varepsilon_0 c^3}\int_{t_1}^{t_2} dt\mathbf{v}(t)\bullet\ddot{\mathbf{v}}(t) = -\int_{t_1}^{t_2} dt\mathbf{v}(t)\bullet\mathbf{F}_{react}$$

  – Thus there is a force that recover the time integrated power

$$\mathbf{F}_{react} = \frac{q^2}{6\pi\varepsilon_0 c^3}\ddot{\mathbf{v}}(t)$$

  – This does not imply that it represents the instantaneous force!

- The equation of motion under the influence of a self-force $\mathbf{F}_0(t)$

$$m\left[\dot{\mathbf{v}}(t) - \tau\ddot{\mathbf{v}}(t)\right] = \mathbf{F}_0 \ , \ \ \tau = \frac{q^2}{6\pi\varepsilon_0 c^3} = \frac{2Z^2}{3}\frac{r_{ele}}{c} \approx 10^{-24}\times Z^2\ [\mathrm{s}]$$

  – this is the *Abraham-Lorentz equation of motion*

  – Note: the force has a time scale τ, which is roughly the time it takes for light to travel across the classical radius of the electron, r_{ele}

- But there are serious problems with the Abraham-Lorentz equation!

Properties of the Abraham-Lorentz equation of motion

- First, there's a run-away solution possible in absence of any force

$$m\left[\dot{\mathbf{v}}(t) - \tau\ddot{\mathbf{v}}(t)\right] = 0 \;\Rightarrow\; \dot{\mathbf{v}}(t) \sim \begin{cases} 0 \\ e^{t/\tau} \end{cases}$$

run-away!

- Rewrite the Abraham-Lorentz equation to avoid the run-away solution

$$m\dot{\mathbf{v}}(t) = \int_0^\infty \mathbf{F}_0(t+\tau x)e^{-x}dx$$

  - This equation has no run-away solutions
- However it includes *pre-acceleration*
  - the force is evaluated in *future* times $t+\tau x$
  - the particle responds to the force before the force is applied
  - this is *NOT* causal model!!
  - however, the time scale of pre-acceleration is tiny: $\tau = \frac{2Z^2}{3}\frac{r_{ele}}{c}$