System planning VT13

- Lecture L17: **Monte Carlo**
- Content:

  - Variance reduction technique
  1. Complementary random numbers
  2. Control variates

Variance reduction technique

Repetition of simple sampling:

- Objective: Get an accurate estimate of E[X]
- The expected value of the estimate is the same as the expected value of X: $E[m_X] = E[X]$.
- The variance of the estimate, Var[m_X], is interesting because it states how much an estimate might deviate from the true value:

$$Var[m_X] = \frac{Var[X]}{n}$$

=>The more observations/samples we use the more likely is it that we get an accurate result.

Variance reduction technique

- Do we have to increase the number of samples to get a more accurate estimation (decreasing Var[m_X])?

  No, we can choose our samples in a smarter way by using variance reduction techniques!

Variance reduction technique

- This course includes three variance reduction techniques:
  1) Complementary random numbers
  2) Control variates
  3) Stratified sampling
- Note that these reduction techniques can be combined!!

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  1. **Complementary random numbers**
  2. Control variates

Complementary random numbers

- A simple variance reduction technique
- Can be applied to all types of Monte Carlo simulations
- Is based on that the generated random numbers are correlated in a ”good” way
- Less samples are necessary to get a good spread of the samples

Complementary random numbers

- Do you remember these formulas for expected value and variance?

$E[aX] = a\,E[X]$, where a is a constant

$Var[X+Y] = Var[X] + Var[Y] + 2\,Cov[X,Y]$

$Var[aX] = a^2\,Var[X]$, where a is a constant

Complementary random numbers

- Let m_{X1} and m_{X2} be two estimates of the expected value for the same stochastic variable, *X*.

$$E[m_{X1}] = E[m_{X2}] = E[X] = \mu_X$$

- Let the average of m_{X1} and m_{X2} be a new estimate:

$$E\left[\frac{m_{X1}+m_{X2}}{2}\right] = \frac{1}{2}E[m_{X1}] + \frac{1}{2}E[m_{X2}] = \mu_X = \underline{E[X]}$$

Since the average of m_{X1} and m_{X2} has the same expected value as E[X] it is a new estimation of E[X]

Complementary random numbers

- How accurate is our new estimate of E[X]? Study the variance!

$$Var\left[\frac{m_{X1}+m_{X2}}{2}\right]=\frac{1}{4}Var[m_{X1}]+\frac{1}{4}Var[m_{X2}]+\frac{1}{2}Cov[m_{X1},m_{X2}]$$

- If m_{X1} and m_{X2} are estimates from two different simulations where simple sampling have been used each with n samples then:

$$Var[m_{X1}]=Var[m_{X2}],\ Cov[m_{X1},m_{X2}]=0\Rightarrow$$

$$Var\left[\frac{m_{X1}+m_{X2}}{2}\right]=\frac{Var[m_{X1}]}{2}<Var[m_{X1}]$$

A more accurate estimate!

Complementary random numbers

- BUT we can do even better if we let m_{X1} and m_{X2} be correlated

$$Var\left[\frac{m_{X1}+m_{X2}}{2}\right]=\frac{1}{4}Var[m_{X1}]+\frac{1}{4}Var[m_{X2}]+\frac{1}{2}Cov[m_{X1},m_{X2}]$$

Can be negative!

- We want the covariance for the estimates to be **negative** which corresponds to that m_{X1} and m_{X2} are negatively correlated.

  $\Rightarrow$ Use complementary random numbers!

Complementary random numbers

Let *U* be *U(0,1)*, and *U** = *1-U*
$\Rightarrow$ U are *U** negatively correlated

Let *Y* = random variable from an arbitrary probability distribution that has been obtained by transforming *U*

Let *Y** = random variable from an arbitrary probability distribution that has been obtained by transforming *U**

Y and *Y** will then also become negatively correlated!

Complementary random numbers

- Let X=g(Y) and X*=g(Y*) then X are X* negatively correlated
- Let m_{X1} be an estimate based on observations of X and let m_{X2} be an estimate based on observations of X*

  $\Rightarrow$ m_{X1} and m_{X2} negatively correlated!!!

Thus:

$$m_{X1}=\frac{1}{n}\sum_{i=1}^{n}x_i,\quad m_{X2}=\frac{1}{n}\sum_{i=1}^{n}x_i^*\Rightarrow$$

$$\frac{m_{X1}+m_{X2}}{2}=\frac{1}{2n}\sum_{i=1}^{n}\left(x_i+x_i^*\right)$$

Complementary random numbers

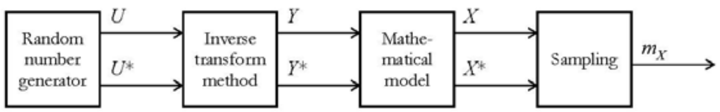
1. Generate n observations of U; $u_1, \ldots, u_n$
2. Calculate their complement $U^* = 1 - U$; $u_1^*, \ldots, u_n^*$
3. Transform U's observations to observations on Y; $y_1, \ldots, y_n$
4. Transform U^*'s observations to observations on Y^*; $y_1^*, \ldots, y_n^*$
5. Instead of considering it as two series each with n observations we can add our observations together to one series with 2n observations
6. Then we use the formula for simple sampling! (we are allowed to do this even though they are not independent)

Complementary random numbers

- The original observations and the complementary observations of K scenario parameters can be combined to 2^K scenarios.
- Example: (K=2)

$\overline{G}, D$	Original scenario
$\overline{G}, D^*$	Complementary scenario
$\overline{G}^*, D$	Complementary scenario
$\overline{G}^*, D^*$	Complementary scenario

Complementary random numbers



- The inverse transform method reduces the correlation between *U* and *U** so that the negative correlation between *Y* and *Y** are not as strong
- Depending on g the correlation between *X* and *X** could become even weaker

Complementary random numbers

- If there is no negative correlation between the result variables $x = g(y)$ and $x^* = g(y^*)$ there is no point of doing complementary random numbers!
- Unnecessary work to apply complementary random numbers on the load in each area in the multi-area model. Apply complementary random numbers on the total load and then distribute it to the areas according to:

$$D_n = \frac{D_{tot}}{\sum_{m \in N} D_m'} D_n'$$

where $D_{tot} = D_1 + D_2 + \ldots D_n$ and *n* number of areas. See example 6.25.

Example 6.22:

A small village with their own grid. Data according to:

- Load is normally distributed with the mean 180 kW and standard deviation 40 kW.
- Three power plants:
  - Hydro plant (no reservoir), installed capacity 150 kW, operation cost 0 SEK/kWh
  - Diesel generator 100 kW, operation cost 1 SEK/kWh, never operated at less than 40 kW. If it is necessary a water heater can be used to consume any surplus generation.
  - Diesel generator 50 kW, operation cost 2 SEK/kWh
  - All power plants are 100% available
- **Calculate ETOC, using Monte Carlo (10 obs. are enough for the estimation) and exactly.**

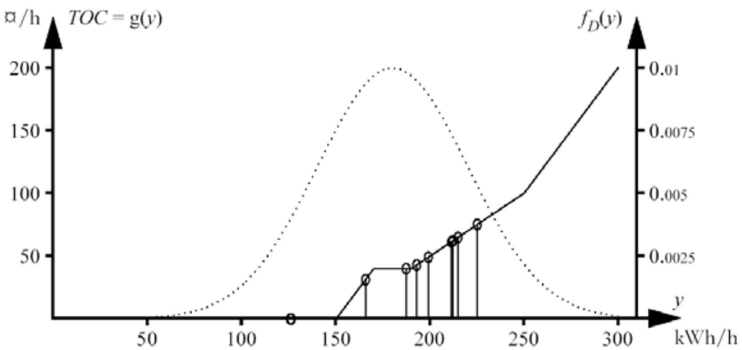
Example 6.22:

- The table shows how the power plants are operated at different load levels, hence the function *g(D)=TOC.*

D [kW]	TOC [SEK/h]	Kommentar
0-150	0	Hydro plant is running
150-170	2*(D-150)	Hydro plant is running on max, the small generator follow the load
170-190	40	Hydro plant is running on max, the large generator is on 40 kW, surplus to heating
190-250	1*(D-150)	Hydro plant is running on max, the large generator follow the load
250-300	100+2*(D-250)	Hydro plant is running on max, the large generator is on max, the small generator follow the load
>300	200	All power plants are on max, load have to be disconnected

Complementary random numbers

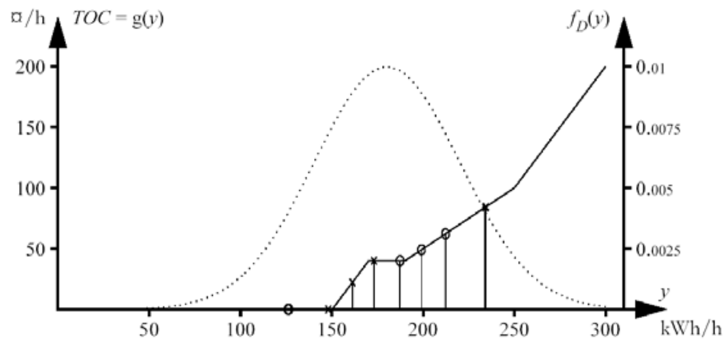
Example (simple sampling, *without complementary random numbers*):



—Cost function (same as in the table)
.... Probability distribution for the load
o Randomized samples (scenarios)

Complementary random numbers

Example (simple sampling, *with complementary random numbers*):



—Cost function (same as in the table)
.... Probability distribution for the load
o Randomized samples (scenarios)
× Randomized samples based on complementary random numbers

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  2. **Control variates**

Control variates

- One of the most important reasons for using Monte Carlo simulation is that it might be too complicated to solve the problem analytically. However, sometimes we do know some things about the solution, and maybe we can solve a part of the problem analytically.
- Use simplified analytical models to improve the result of the Monte Carlo simulation.
- The idea is to introduce another stochastic variable – a control variate - Z. By estimating the difference X-Z we get a more accurate estimate of E[X].

Control variates

- Assume X is a stochastic variable with an expected value, μ_X, which is **unknown**.
- Let Z be a stochastic variable with an expected value, μ_Z, that is **known**. Z is referred to as the control variate.
- We want to find the estimate of E[X], ie m_X .
  But instead we find an estimate of E[X-Z], ie $m_{(X-Z)}$
- The estimation for E[X] we then calculate as:

$$m_X = m_{X-Z} + \mu_Z$$

known!

since

$$E[m_{(X-Z)} + \mu_Z] = E[X-Z] + \mu_Z = E[X] - E[Z] + \mu_Z =$$

$$= E[X] - \mu_Z + \mu_Z = E[X]$$

Same expected value!

Control variates

Why do we have to involve another stochastic variable, Z, is it not enough with X?

Answer: The estimate $m_{(X-Z)} + \mu_Z$ is more accurate than m_X (even though they are both estimates of the same expected value E[X]!).

$$Var[m_{(X-Z)} + \mu_Z] < Var[m_X]$$

Estimations of E[X]!

Control variates

- Want to show: $Var[m_{(X-Z)} + \mu_Z] < Var[m_X]$
- Start by studying the variance for the difference X-Z:

$$Var[X-Z] = Var[X] + Var[Z] - 2Cov[X,Z]$$

- Choose Z so X and Z are strongly positively correlated

$\Rightarrow$

Cov[X,Z] a positive number $\Rightarrow$ possible to get 2Cov[X,Z] > Var[Z]

$$\Rightarrow Var[X-Z] = Var[X] + \underbrace{Var[Z] - 2Cov[X,Z]}_{\text{Becomes negative!}}$$

$$\Rightarrow \boxed{Var[X-Z] < Var[X]}$$

Control variates

- Want to show: $Var[m_{(X-Z)} + \mu_Z] < Var[m_X]$
- Remember theorem:

> The variance for the estimation when using simple sampling:
>
> $$Var[m_X] = \frac{Var[X]}{n}$$

$$\Rightarrow Var[m_{(X-Z)} + \mu_Z] = Var[m_{(X-Z)}] = \frac{Var[X-Z]}{n} < \frac{Var[X]}{n} = Var[m_X]$$

The variance of a constant is zero!

$Var[X-Z] < Var[X]$

- Thus, $Var[m_{(X-Z)} + \mu_Z] < Var[m_X]$

Control variates

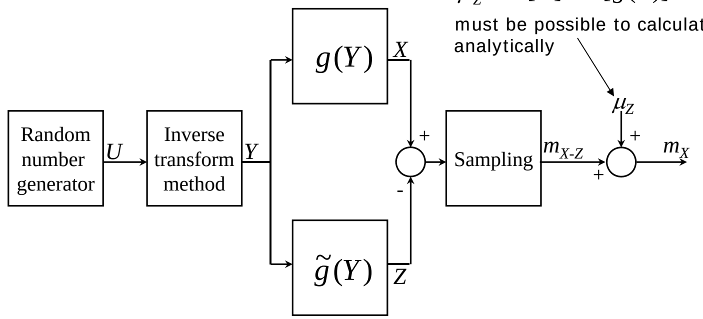
Conclusion:

- The estimate $m_{(X\text{-}Z)}$ + μ_Z is more accurate than m_X (even though they are both estimates of the same expected value.)
- The larger positive correlation between *X* and *Z* the better it is.

Control variates

$\mu_Z = E[Z] = E[\tilde{g}(Y)]$

must be possible to calculate analytically



$g(Y)$ Complicated mathematical model

$\tilde{g}(Y)$ Simplified mathematical model

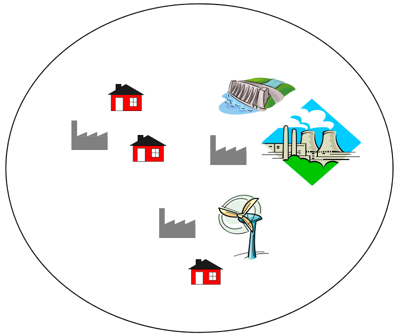
Control variates

- Note that X and Z are results from two models of the **same** system.

Our assumption that X and Z are strongly **positively** correlated seems reasonable!

Control variates

- What is the simplified electricity market model?
  - One area model
  - Same assumptions as before:
    1. How the actors act in the market
    2. Power system



(Note that with one area we do not have any transmission lines ⇒ Do not have to consider transmission limits and losses)

Control variates

- Can be formulated as an optimization problem:

$$\min \sum_{g \in G} \beta_g \tilde{G}_g + \beta_U \tilde{U}$$

$$when \quad \sum_{g \in G} \tilde{G}_g + \tilde{U} = D_{tot}$$

This is the PPC!

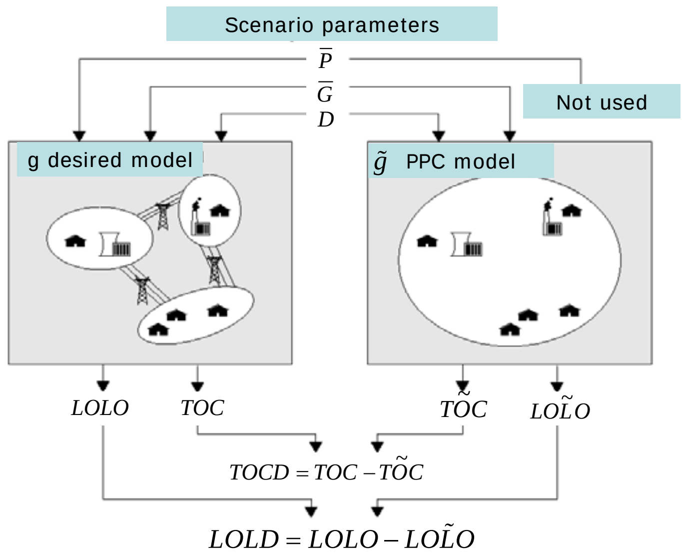
$$0 \le \tilde{G}_g \le \overline{G}_g$$

$$0 \le \tilde{U}$$

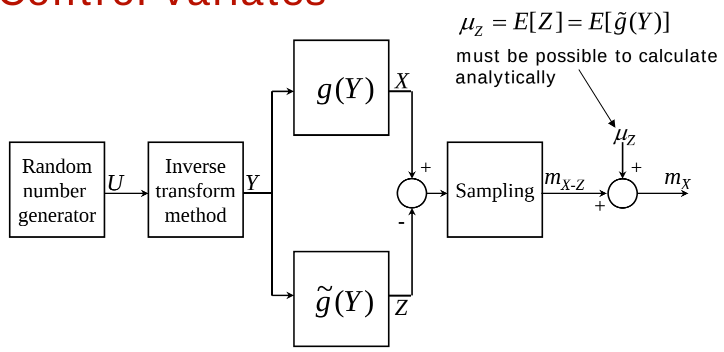
- The solution of the problem gives us the result variables:

$$T\tilde{O}C = \sum_{g \in G} \beta_g \tilde{G}_g$$

$$LO\tilde{L}O = \begin{cases} 1 & \text{om } \tilde{U} > 0 \\ 0 & \text{annars} \end{cases}$$

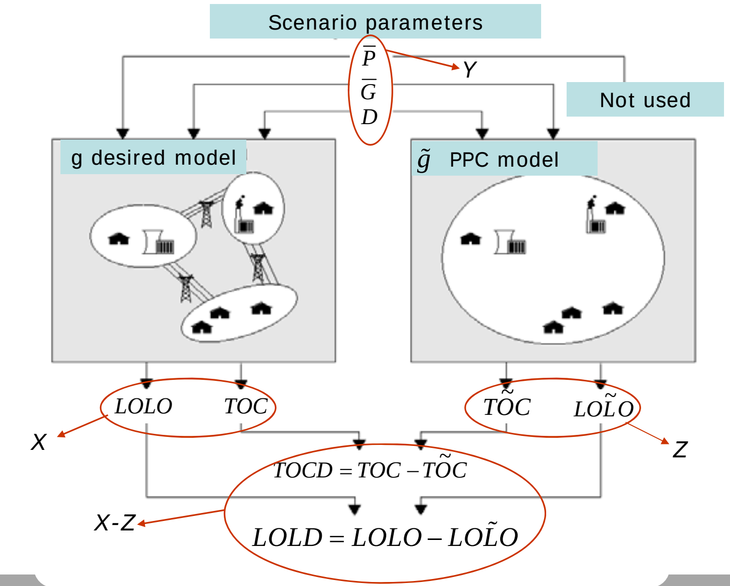


Control variates



$g(Y)$ Complicated mathematical model

$\tilde{g}(Y)$ Simplified mathematical model



Example 6.22:
Three variance reduction techniques

- The table shows how the power plants are operated at different load levels, hence the function *g(D)=TOC*.

How does it look like for PPC???

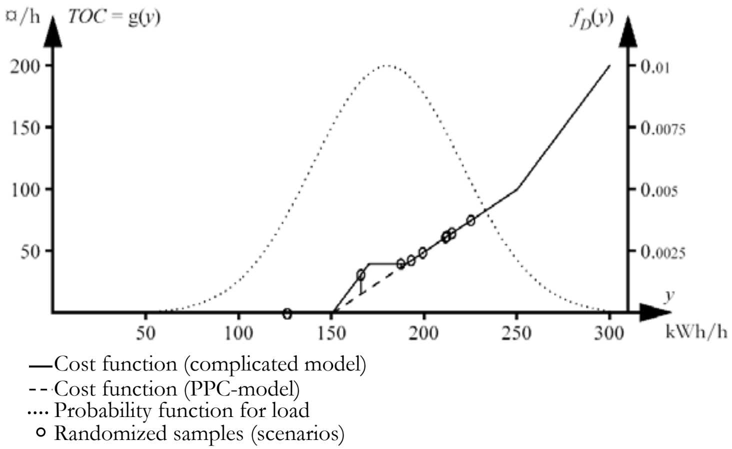
D [kW]	TOC [SEK/h]	Kommentar
0-150	0	Hydro plant is running
150-170	2*(D-150)	Hydro plant is running on max, the small generator follow the load
170-190	40	Hydro plant is running on max, the large generator is on 40 kW, surplus to heating
190-250	1*(D-150)	Hydro plant is running on max, the large generator follow the load
250-300	100+2*(D-250)	Hydro plant is running on max, the large generator is on max, the small generator follow the load
>300	200	All power plants are on max, load have to be disconnected

Control variates

Example

- Same scenarios as before but now the samples are the difference between the complicated and simplified models

Scenario	D [kWh/h]	TOC [¤/h]	$\tilde{TOC}$ [¤/h]	$TOCD$ [¤/h]
1	187.2	40.0	37.2	2.8
2	125.7	0	0	0
3	212.1	62.1	62.1	0
4	126.1	0	0	0
5	199.0	49.0	49.0	0
6	214.8	64.8	64.8	0
7	225.0	75.0	75.0	0
8	211.3	61.3	61.3	0
9	165.6	31.1	15.6	15.6
10	192.8	42.8	42.8	0



Control variates

- Estimation of $E[TOCD]$: m_{TOCD} = 1.84 ¤/h
- If using PPC we get $\mu_{\tilde{TOC}}$ = 36.27 ¤/h
- Thus:

  $m_{TOC} = m_{TOCD} + \mu_{\tilde{TOC}}$ = 38.11 SEK/h

  (Theoretical value 39.66 SEK/h)

Control variates

- Why is the estimation better with control variates?
  - 1) Without control variates:

$$ETOC = E[TOC] = \underbrace{\int_{-\infty}^{\infty} f_D(y)g(y)dy}_{\text{The white and grey area}}$$

  - 2) With control variates:

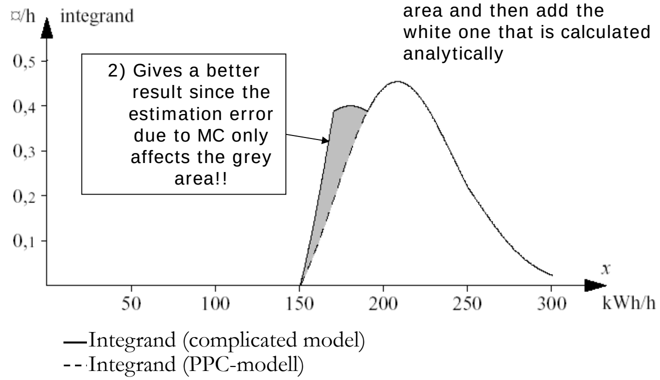
$$ETOC = E[TOCD] + \mu_{\tilde{TOC}} = \underbrace{\int_{-\infty}^{\infty} f_D(x)[g(y) - \tilde{g}(y)]dy}_{\text{The grey area}} + \underbrace{\mu_{\tilde{TOC}}}_{\text{The white area}}$$

Control variates

We want ETOC => white area plus grey area. Two ways to get it:

1) Use MC to get the whole area.

2) Used MC to get the grey area and then add the white one that is calculated analytically



Calculate!

- Exercise 6.15- Control variates

Control variates

- Example 6.15:
- The losses are modeled by a quadratic function.
- Hydro power plant:
  - Installed capacity 400 kW
  - No operation cost
  - 100 % available
- Diesel generators
  - Installed capacity per generator 200 kW
  - 3 generators
  - Operation cost 0.5 kr/kWh

Control variates

- Contin. example
- A Monte Carlo simulation has been carried out with the following results:

$\overline{H}$ [kW]	$\overline{G}_1$ [kW]	$\overline{G}_2$ [kW]	$\overline{G}_3$ [kW]	D
400	200	200	0	724
400	200	200	200	503
400	200	0	200	381
400	200	200	200	612
400	200	200	200	449

- What is the estimation of ETOC when the control variate method is used?

Control variates

- Contin. example
- Know that $ETOC_{PPC}$=70.4 kr/h
- Control variate method:

A) For each scenario:

1. Calculate TOC_i
2. Calculate $TOC_{PPC,i}$
3. Calculate $TOCD_i = TOC_i - TOC_{PPC,i}$

B) The estimation of ETOCD is the average of the observations:

$$ETOCD = \frac{1}{n}\sum_{i=1}^{n} TOCD_i$$

C) Finally:

$ETOC = ETOCD + ETOC_{PPC}$

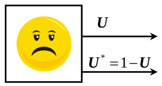
Calculate!

- Example 6.25 - Total load

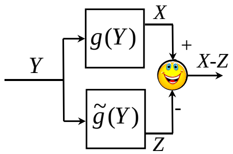
Complementary random numbers

- Example
  - In one electricity market we have two areas.
  - In the first area the load is N(500,40)-distributed and in the second area the load is N(450,30)-distributed.
  - The load levels in both the areas are independent stochastic variables.
  - We want to randomize the total load in the system that is greater than 1000 MW.

The most important from today:



- **Complementary random numbers**
  - Generate negatively correlated numbers: U and U^* = *1-U*
  - ⇒ More accurate estimate of E(X) due to that X and X^* have **negative** correlation



- **Control variate**
  - For the same scenario parameter values solve the
    - multi-area model => X
    - one-area model => Z
  - Estimate the difference X-Z and then add μ_Z (from PPC)
  - ⇒ More accurate estimate of E(X) due to that Z and X have **positive** correlation

Next time:

- Objective: We want a more accurate estimate of E[X] without having to generate more samples!
- Method: Choose our observations smarter, use variance reduction techniques:
  - Complementary random numbers ✓
  - Control variates ✓
  - Stratified sampling
  - Combine them!