System planning VT13

- Lecture L18: **Monte Carlo**
- Content:
  1. **Variance reduction techniques**
     - **Stratified sampling**
  3. Combine variance reduction techniques
  4. Calculate/Hints for home assignment

Stratified sampling

- Divide the samples into different subpopulations referred to as strata.

- Why?
  1. **Practical reasons:**
     - Maybe different parts of the study must be performed using different methods
  2. **Efficiency gains:**
     - Collecting samples with similar properties in the same stratum will reduce the variance in each stratum h.

  $$Var[m_{X_h}] = \frac{Var[X_h]}{n_h}$$

  - Which in turn give us a more accurate estimate of $m_{X,h}$ without needing to use a greater number samples n_h; we estimate $m_{X,h}$ more efficient.
  - The estimate m_X, we get by weighting $m_{X,h}$ for the different strata.

Stratified sampling

- Theory:
  1. Let X be a stochastic variable with the sample space Γ
  2. Divide the sample space in L parts, strata, where stratum h is the subset $\Gamma_h \subset \Gamma$. Strata may not overlap $\Gamma_h \cap \Gamma_k = \varnothing, \ \forall \ h \neq k$
  3. Each stratum has a stratum weight

  $$\omega_h = \frac{N_h}{N} = P(X \in \Gamma_h)$$

  ω_h = stratum weight for stratum h
  N_h = number of possible outcomes in stratum h
  N = number of possible outcomes in the entire population

Stratified sampling

- Analyze each stratum separately:
  4. L separate stochastic variables, X_h, h=1,…,L, with the sample space Γ_h
  5. If possible determine

  $$E[X_h] = \mu_{X_h}$$

  analytically
  6. Else do simple sampling $m_{X_h} = \frac{1}{n_h} \sum_{i=1}^{n_h} x_{h,i}$

  m_{X_h} = estimate of the expected value in stratum h

  $x_{h,i}$ = *value of the i*:th sample from stratum h

  n_h = number of samples in stratum h

Stratified sampling

7. Weight the estimates for the different strata to get the estimation of E[X]:

$$m_X = \sum_{h=1}^{L} \omega_h m_{X_h}$$

8. The variance for the estimation is m_X:

$$Var[m_X] = \sum_{h=1}^{L} \omega_h^2 \frac{Var[X_h]}{n_h}$$

> Note that how you choose strata and n_h affect the variance:
> – With a good choice the variance is reduced compared to simple sampling. BUT with a bad choice the variance can be increased!

Stratified sampling

- To conclude:
  1. Divide the samples into L subpopulations- *strata*.
  2. Estimate the expected value of X in each stratum h, m_{Xh}
  3. Weight the estimates for each stratum according to the size of the stratum

$$m_X = \sum_{h=1}^{L} \omega_h m_{X_h}, \text{ where } \omega_h = P(X \in \Gamma_h)$$
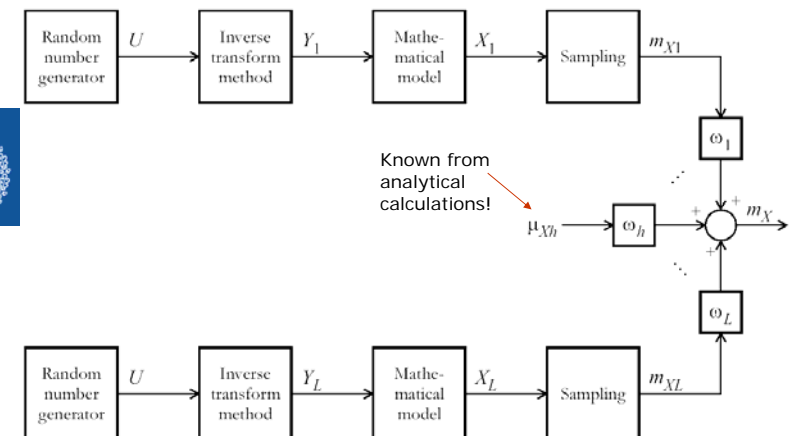
Stratified sampling

- Why can the estimate be more accurate using stratified sampling?

  – It is possible to identify homogenous strata. The samples for homogenous strata are very similar which makes the variance for these strata small. The expected values for homogenous strata can be estimated with high accuracy without having to generate so many samples (why take more of something that is very similar to the previous ones?). We say that we estimate the expected value more efficient.

  – INSTEAD we can generate more samples in strata where they do more good, ie in strata in which the samples are very different.

Stratified sampling

Known from analytical calculations!

Stratified sampling
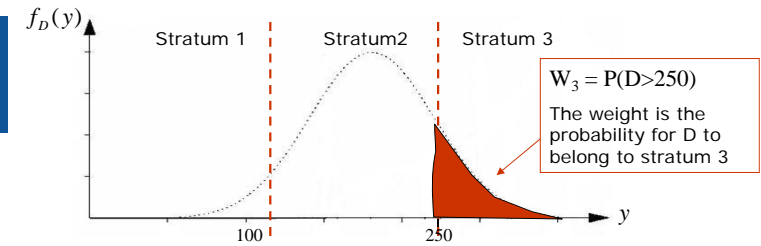
- Definition of stratum:

  A stratum is defined by intervals of the scenario parameters.

Stratified sampling

- For example for the load, *D*, that is a scenario parameter in electricity market simulations the strata can be defined as:

$f_D(y)$

Stratum 1   Stratum2   Stratum 3

$W_3 = P(D>250)$

The weight is the probability for D to belong to stratum 3

100   250   *y*

- Thus we have to be able to generate random numbers from a certain interval of D (see appendix E)

Stratified sampling

- Two important decisions that are vital for how accurate our estimation is are:

  1. Definition of stratum
  2. Definition of number of samples in each stratum

Stratified sampling

1. Definition of stratum

- For stratified sampling to be efficient the scenarios that belong to the same stratum should give similar results. Then the variance for each strata is as small as possible!

- The trick is to identify scenario types (intervals for D, etc) that give similar values for *TOC* and *LOLO*.

Stratified sampling

1. Definition of stratum

To define our stratum we need a strata tree.
A strata tree has the following properties:

- Each node (except the root) specifies a subset of the sample space for one or more scenario parameters.
- Each node has a certain node weight = the probability that the scenario parameter belongs to the specified subset.
- Each branch of the tree should specify a part of the sample space for each of the scenario parameters in the model.

Stratified sampling

1. Definition of stratum

- All nodes do not have to specify a subset of scenario parameters. As long as you can calculate the node weights you are welcome to introduce help variables.
- The children of a certain node must define subsets for the same scenario parameters
- The sum of the node weights of the children must always be exactly 1

Stratified sampling

1. Definition of stratum

- The idea with a strata tree:
  - Put all possible outcomes of maximal available capacities in nodes
  - Below these nodes put the different load nodes
- Thus: "Production nodes" are in the top of the tree and "load nodes" are in the bottom of the tree
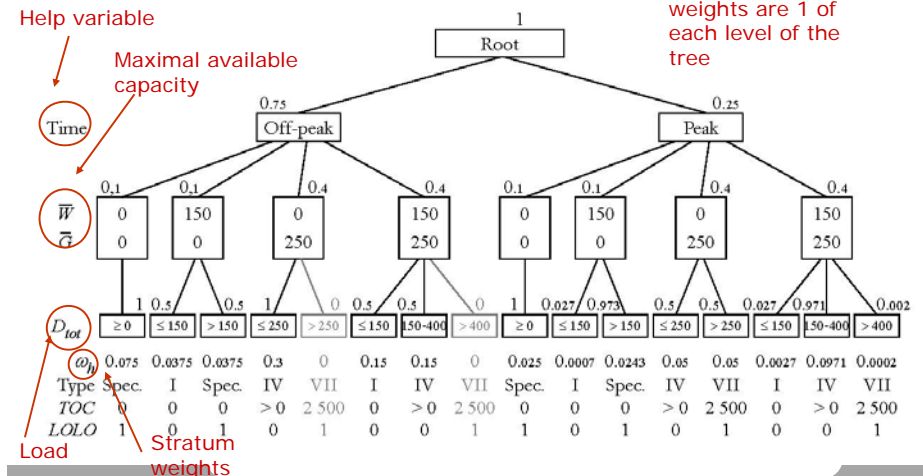

Stratified sampling

NOTE! The sum of the children's weights are 1 of each level of the tree

Help variable

Maximal available capacity

Load

Stratum weights

Stratified sampling

- Two important decisions that are vital for how accurate our estimation is are:

  1. Definition of stratum
  2. **Definition of number of samples in each stratum**

Stratified sampling

2. <u>Definition of the number of samples in each stratum</u>
- Use Neyman allocation:
  – We want to distribute n samples on L strata. It has been shown that $Var[m_x]$ is minimized if the samples are distributed according to:

$$n_h = n \frac{\omega_h \sigma_{X_h}}{\sum_{i=1}^{L} \omega_i \sigma_{X_i}}$$

$\sigma_{X_h} = \sqrt{Var[X_h]}$ Is the standard deviation for stratum h

---

Stratified sampling

<u>2. Definition of the number of samples in each stratum</u>

- But σ_{X_h} is not unknown! Use the estimate of the standard deviation instead.

$$s_{X_h} = \sqrt{\frac{\sum_{i=1}^{n_h}(x_{hi} - m_{X_h})^2}{n_h}}$$

- But to calculate this estimation we need some samples of X_h!

Stratified sampling

<u>2. Definition of the number of samples in each stratum</u>

- Thus: To allocate according to Neyman you must have samples. Samples you get when having done simulations. BUT to run simulations you need to allocate…. Problem!

- Solution: Impossible to use Neyman when you start to simulate. Divide the simulation in parts.
  – Part 1: Run a predefined number of samples for each strata (can be different for different strata)
  – Use the result in part 1 to do a Neyman allocation
  – Run a new simulation.

Stratified sampling

2. Definition of the number of samples in each stratum

- Another problem....
  - Neyman allocation use the variance of ONE stochastic variable. What if we want to study both, *TOC* and *LOLO? What do we do?*
  - Solution: Do allocation for both TOC and LOLO and use the average of these allocations.

Stratified sampling

2. Definition of the number of samples in each stratum

- Another problem...
  - What if we have taken too many samples from some strata in part 1 of the simulation? (we have stolen samples from the other strata...). What do we do?
  - Solution: Adjust the allocation of the samples for part 2 of the simulation.

System planning VT13

- Lecture L18: **Monte Carlo**
- Content:
  1. Variance reduction techniques
     - Stratified sampling
  3. **Combine variance reduction techniques**
  4. Calculate/Hints for home assignment

Combine variance reduction techniques

- It is perfectly possible to combine the three variance reduction techniques! Example (ETOC):

  - Calculate $ETOC_{PPC}$ (possible to calculate analytically)
  - For each strata (stratified sampling):
    - Randomize the loads D and the complementary loads D^* (complementary random numbers)
    - Calculate TOC and $TOC_{PPC} \Rightarrow TOCD$, and TOC^* and $TOC_{PPC}^* \Rightarrow TOCD^*$ (control variates)
    - Estimate ETOCD based on the observations of TOCD and $TOCD^*$
  - Estimate $ETOCD$ by weighting the $ETOCD_h$ together (use the strata weights)
  - $ETOC = ETOC_{PPC} + ETOCD$

System planning VT13

- Lecture L18: **Monte Carlo**
- Content:
  1. Variance reduction techniques
     - Stratified sampling
  3. Combine variance reduction techniques
  4. Calculate/Hints for home assignment

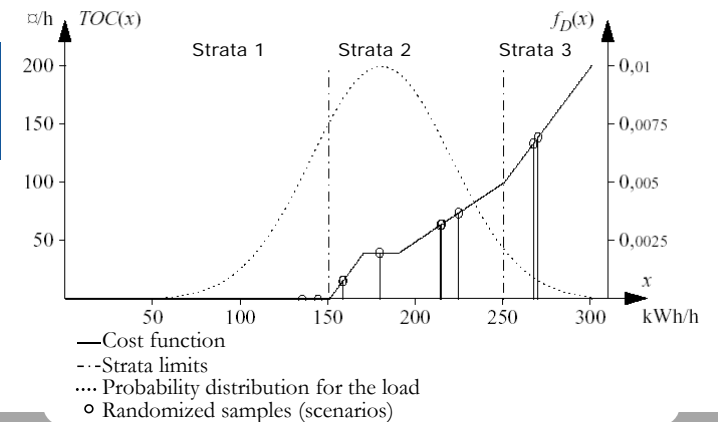
Calculate!

- Example 6.22

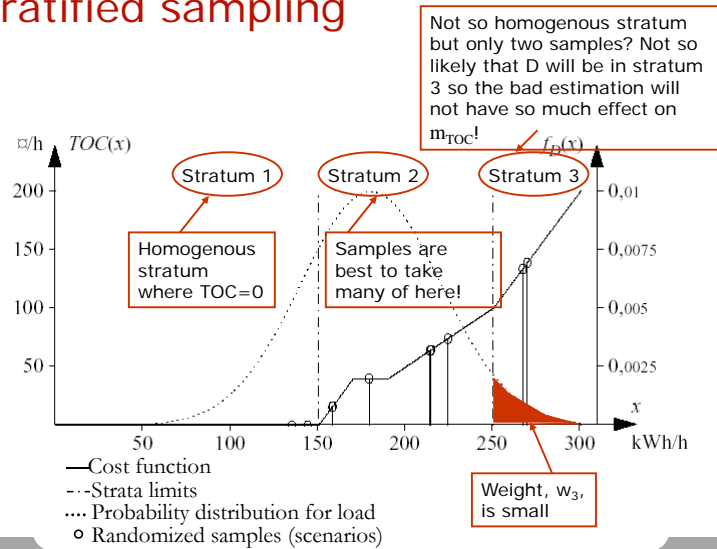# Stratified sampling:

Example 6.28: (Same system as in 6.22)

A small village with their own grid. Data according to:
- Load is normally distributed with the mean 180 kW and standard deviation 40 kW.
- Three power plants:
  - Hydro plant (no reservoir), installed capacity 150 kW, operation cost 0 SEK/kWh
  - Diesel generator 100 kW, operation cost 1 SEK/kWh, never operated at less than 40 kW. If it is necessary a water heater can be used to consume any surplus generation.
  - Diesel generator 50 kW, operation cost 2 SEK/kWh
  - All power plants are 100% available
- **Calculate ETOC, using Monte Carlo (10 obs. are enough for the estimation) and exactly.**

Stratified sampling

Genererate 2 samples for Stratum 1 and Stratum 3 and 6 samples from Stratum 2. Predefined number...
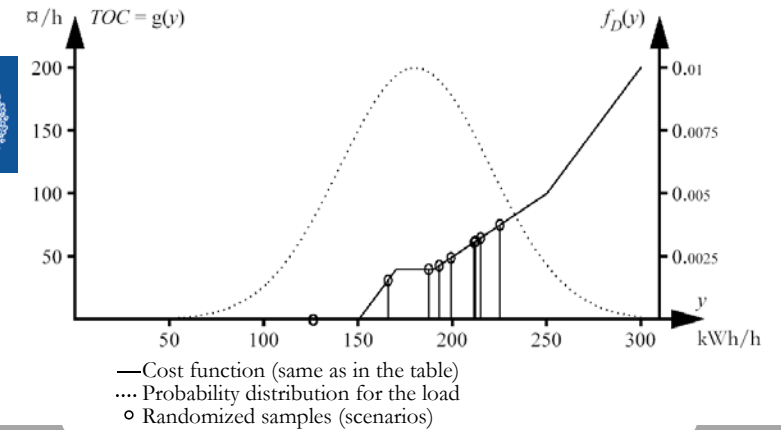

— Cost function
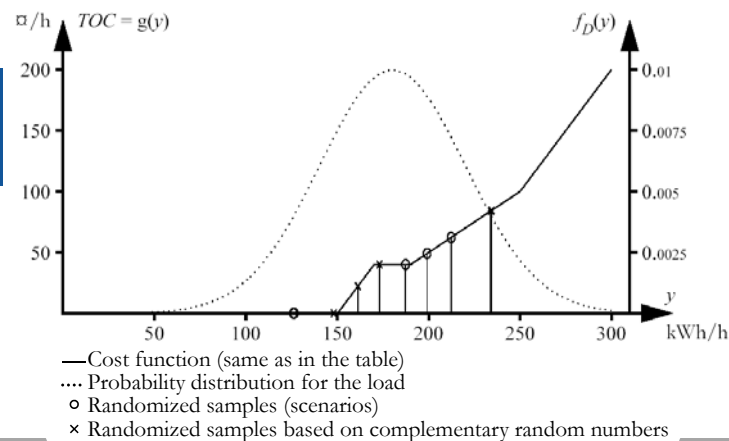-·- Strata limits
···· Probability distribution for the load
○ Randomized samples (scenarios)

Stratified sampling

Not so homogenous stratum but only two samples? Not so likely that D will be in stratum 3 so the bad estimation will not have so much effect on m_{TOC}!

Stratum 1   Stratum 2   Stratum 3

Homogenous stratum where TOC=0

Samples are best to take many of here!

Weight, w_3, is small

—Cost function
-·-Strata limits
···· Probability distribution for load
○ Randomized samples (scenarios)

29

Simple sampling

Example (simple sampling, *without complementary random numbers*):

—Cost function (same as in the table)
···· Probability distribution for the load
○ Randomized samples (scenarios)

30

Complementary random numbers

Example (simple sampling, *with complementary random numbers*):

—Cost function (same as in the table)
···· Probability distribution for the load
○ Randomized samples (scenarios)
× Randomized samples based on complementary random numbers

31

Control variates

—Cost function (complicated model)
-- Cost function (PPC-model)
···· Probability function for load
○ Randomized samples (scenarios)

32
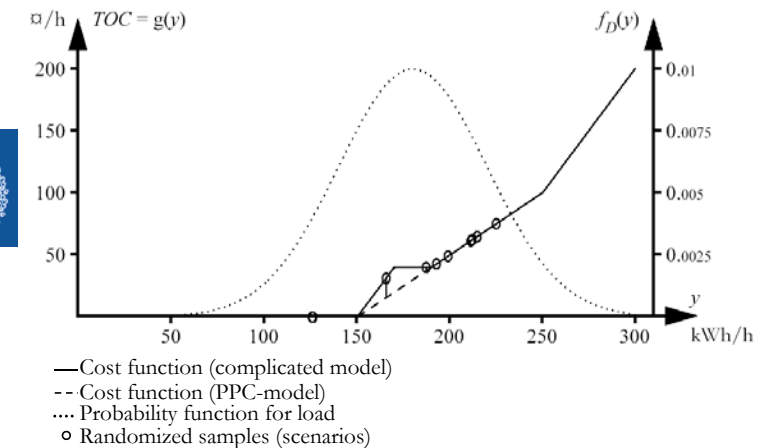
Control variates

- Why is the estimation better with control variates?
  - 1) Without control variates:

$$ETOC = E[TOC] = \int_{-\infty}^{\infty} f_D(y)g(y)dy$$

  $\underbrace{}$ The white and grey area

  - 2) With control variates:

$$ETOC = E[TOCD] + \mu_{T\tilde{O}C} = \int_{-\infty}^{\infty} f_D(x)\big[g(y) - \tilde{g}(y)\big]dy + \mu_{T\tilde{O}C}$$

  $\underbrace{}$ The grey area $\quad\underbrace{}$ The white area

Control variates

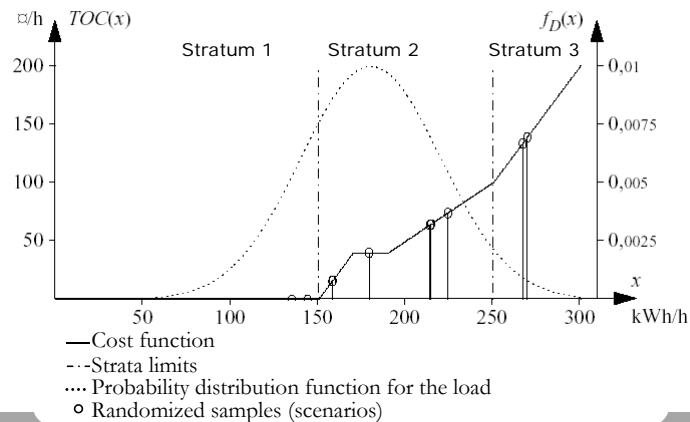We want ETOC => white area plus grey area. Two ways to get it:

1) Use MC to get the whole area.

2) Used MC to get the grey area and then add the white one that is calculated analytically
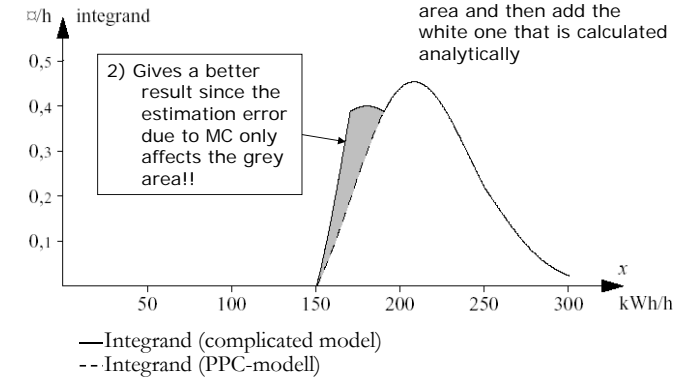
2) Gives a better result since the estimation error due to MC only affects the grey area!!

—Integrand (complicated model)
- - Integrand (PPC-modell)

Example:
Three reduction techniques

Stratified sampling:

—Cost function
- - Strata limits
.... Probability distribution function for the load
o Randomized samples (scenarios)

Example:
Three reduction techniques

- Results for the estimation of ETOC:
  - Exact value (calculated theoretically): 39.66 SEK/h
  - Simple sampling: 41.61 SEK/h (diff 1.95)
  - Complementary random numbers: 38.13 SEK/h (diff 1.53)
  - Control variates: 38.11 SEK/h (diff 1.55)
  - Stratified sampling: 38.99 SEK/h (diff 0.67)

Improved result!

Calculate!

• Hints for home assigment

Hints for home assignment

• Examples in the compendium that are especially interesting for passing home assignments:
  – For 17) and 18): Example 6.8
  – For 21): Example 6.1 and 6.2
  – For 23): Example 6.30 b)
    • Calculate the maximal loss, this matters for the definition of the strata limits, see figure 6.28.
    • Think about which power plants that will be operate for different load levels (only hydro and thermal available NOT WIND).
  – For 24) See slide 37 "Combine variance reduction techniques" and example 6.29.
• Note that Matlab's function "norminterval" also can return the complementary scenarios, write "help norminterval" in the Matlab command window and you will find about information about the function.
• Matlab's function "normcdf" calculates the $P(X<=x)$ when X is normally distributed.
• Do NOT forget to motivate how you calculate the strata weights!

Recommended exercise on stratified sampling

• Exercise 6.16 (has solution in the compendium)

Only repetition left!

• Repetition:
  Friday March 8
  10-12, **V1**

  Mikael gives hints for the exam!