Solutions to Exam in EL2745 Principles of Wireless Sensor Networks, June 08, 2012

1. (a)

Consider a B-bit converter producing integers $\{0,1,\dots 2^{B-1}\}$. The quantization interval for a signal with amplitude $[-A,A]$ is

$$\Delta = \frac{A-(-A)}{2^B} = A2^{1-B}$$

since D/A convertor recovers the signal by assigning the values to the halfway of each interval Δ then the quantization error ε can get any value in the range $[0,\Delta/2]$. So ε has a uniform distribution $\mathcal{U}(0,\Delta/2)$ with the pdf $f(x) = 2/\Delta$ for $x \in [0,\Delta/2]$.

(b) One can compute the root mean square rms$(g(t))$ of the signal $g(t)$ on the period $[0,T]$ as following

$$\text{rms}(g(t)) = \sqrt{\frac{1}{T}\int_0^T [g(t)]^2 dt}.$$

Given Δ and knowing the interval of error $[0,\Delta/2]$ the value of rms of the quantization error is

$$\text{rms}(\text{error}) = \sqrt{\frac{2}{\Delta}\int_0^{\frac{\Delta}{2}} \varepsilon^2 d\varepsilon} = \sqrt{\frac{\Delta^2}{12}}.$$

(c) the signal to noise ratio resulting after the quantization is as following

$$\text{SNR} = \frac{\text{signal power}}{\text{error power}} = \frac{\text{rms}(s(t))}{\sqrt{\frac{\Delta^2}{12}}},$$

where rms$(s(t))$ for sinusoid signal with amplitude $[-A,A]$ is obtained as

$$\text{rms}(s(t)) = \sqrt{\frac{1}{T}\int_0^T [A\sin(\frac{2\pi t}{T})]^2 dt} = \frac{A}{\sqrt{2}}.$$

Hence, SNR reads as

$$\text{SNR} = \frac{\frac{A}{\sqrt{2}}}{\sqrt{\frac{\Delta^2}{12}}} = \frac{\frac{A}{\sqrt{2}}}{\frac{A2^{1-B}}{\sqrt{12}}} = \sqrt{6}2^{B-1} = 6B + 10\log(1.5)\text{dB} \approx 6B + 1.76.$$

(d) By solving SNR $\approx 6B + 1.76 < 60$ for B, we conclude $B = 10$ bits.

(e) A 16-bit A/D converter yields a SNR of $6 \times 16 + 1.76 = 97.76$dB.

2. (a)

That the packet is correctly received after n transmissions is to say that there are $n-1$ corrupted transmissions and 1 correct transmission. The probability is thus given by

$$P_n = (1-P_e)P_e^{n-1}\,.$$

The average number of transmissions required for a correct reception is

$$\begin{aligned} N &= \sum_{n=0}^{\infty} nP_n = \sum_{n=1}^{\infty} n(1-P_e)P_e^{n-1} \\ &= (1-P_e)\frac{d}{dP_e}\sum_{n=1}^{\infty} P_e^n \\ &= (1-P_e)\frac{d}{dP_e}\left(\frac{P_e}{1-P_e}\right) = \frac{1}{1-P_e} \end{aligned}$$

(b) Since there are a total number of N data packet transmission and $N-1$ acknowledgement packet transmissions, the average delay experienced by a packet is

$$D = \left(\frac{L}{R} + t_d\right)N + \left(\frac{L_{\text{ARQ}}}{R} + t_d\right)(N-1)\,,$$

where N is the average number of transmissions from (a).

(c) The probability of error, P_e, in this context is the probability that at least someone out of $n-1$ nodes transmits given that one is already transmitting. So $P_e = 1-(1-p)^{n-1}$.

(d) We define the function $P : \mathbb{R}^2 \to \mathbb{R}$ as

$$P(p,n) := \Pr(\text{success}) = n \cdot p(1-p)^{n-1}\,.$$

For a fixed p, $P(p,n)$ is monotone increasing for $n \leq -1/\ln(1-p)$ and monotone decreasing for $n \geq -1/\ln(1-p)$ and therefore $P(p,n)$ is minimized either at $n = A$ or at $n = B$ for $n \in [A,B]$. Therefore, we have to find

$$p^\star = \arg\max_p(\min\{P(p,A), P(p,B)\})\,.$$

For a fixed n, $P(p,n)$ is monotone increasing for $p \leq 1/n$ and monotone decreasing for $p \geq 1/n$ (for $p \in [0,1]$). Furthermore, $P(1/A,A) \geq P(1/A,B)$ and $P(1/B,B) \geq P(1/B,A)$ for $B \geq A+1$ and therefore the intersection between $P(p,A)$ and $P(p,B)$ is between the maxima of $P(p,A)$ and $P(p,B)$, respectively. Thus $p^\star$ is found where $P(p^\star,A) = P(p^\star,B)$. Therefore,

$$A \cdot p^\star \cdot (1-p^\star)^{A-1} = B \cdot p^\star \cdot (1-p^\star)^{B-1}$$

$$\frac{A}{B} = (1-p^\star)^{B-1-(A-1)} = (1-p^\star)^{B-A}$$

$$p^\star = 1 - \sqrt[B-A]{\frac{A}{B}}$$

3. (a)

   If we consider $k \geq 0$ and the intermediate nodes are equidistant, then the sum of required energy between source and destination considering k intermediate node is as follows

   $$E = (k+1)\left(\frac{d}{k+1}\right)^{\alpha} = d^{\alpha}(k+1)^{1-\alpha}.$$

   (b) since $1-\alpha < 0$ if k increases E decreases and in limit point, when k goes to infinity E goes to zero. So there is no optimal number of nodes to minimize the energy consumption (infinity number of nodes here makes the energy to be zero).

   (c) A new model of energy consumption with the constant value is more realistic: $E(A,B) = d(A,B)^{\alpha} + C$. If we put this new formula in the limit computed in part (a), the minimum required energy for transmission would be a value greater than zero and it is more reasonable, because in real world it is impossible to send data without any energy consumption.

   (d) We have

   $$E = (k+1)\left(\frac{d}{k+1}\right)^{\alpha} + (k+1)C.$$

   By taking the derivative

   $$\frac{dE}{dk} = \left(\frac{d}{k+1}\right)^{\alpha} - \left(\frac{d}{k+1}\right)^{\alpha}\alpha + C = \left(\frac{d}{k+1}\right)^{\alpha}(1-\alpha) + C$$

   and putting it to zero, we have:

   $$k = d\left(\frac{\alpha-1}{C}\right)^{\frac{1}{\alpha}} - 1$$

   which is the optimal number of intermediate nodes that minimizes the overall energy consumption.

   (e) If we put the computed value of k in previous case into the energy consumption equation (of previous section), the following closed form can be achieved:

   $$E(S,T) = (k+1)\left(\frac{d}{k+1}\right)^{\alpha} + (k+1)C$$

   $$k = d\left(\frac{\alpha-1}{C}\right)^{\frac{1}{\alpha}} - 1$$

so

$$E(S,T) = \alpha d \left(\frac{C}{\alpha - 1} \right)^{\frac{\alpha-1}{\alpha}} .$$

4. (a) Let x be the measured signal, which is given by the source's transmitted signal plus the measurement noise. Denote the false alarm as $\Pr(x < 0, D = H_0)$, and miss detection as $\Pr(x > 0, D = H_1)$, where x is the signal and D is the decision made by per every node. For the false alarm, we have

$$\begin{aligned}\Pr(x < 0, D = H_0) =& \Pr(x < 0) \Pr(D = H_0 | x < 0) \\ =& \frac{1}{2} \int_{-\infty}^{0} \int_{-x}^{\infty} \frac{1}{\sigma_n} e^{-\frac{y^2}{2(\sigma_n^2 + \sigma_S^2)}} dy dx .\end{aligned}$$

Similarly, we have

$$\begin{aligned}\Pr(x > 0, D = H_1) =& \Pr(x > 0) \Pr(D = H_1 | x > 0) \\ =& \frac{1}{2} \int_{0}^{\infty} \int_{-\infty}^{-x} \frac{1}{\sigma_n} e^{-\frac{y^2}{2(\sigma_n^2 + \sigma_S^2)}} dy dx .\end{aligned}$$

(b) Let r_i be the distance from the source to a node, and let S be the signal power. Then for node 1 to be involved in a decision rather than some other node one must have

$$\frac{S}{10 r_1^2} > \frac{S}{r_i^2} \Rightarrow \frac{r_i}{r_1} > \sqrt{10}$$

For reasons of symmetry, we need only consider one of the nodes 2-6 with node 1 as the origin. Without loss of generality, let the source at position (x,y) and consider the equal SNR respecting to node 3. Then we have

$$\frac{r_3^2}{r_1^2} = \frac{(x-d)^2 + y^2}{x^2 + y^2} = 10 \Rightarrow \frac{(x + \frac{1}{9}d)^2}{(\frac{\sqrt{10}}{9}d)^2} + \frac{y^2}{(\frac{\sqrt{10}}{9}d)^2} = 1 ,$$

which is an ellipse with center in $(-d/9, 0)$ having x- and y-axis radius $(d\sqrt{10}/9, d\sqrt{10}/9)$ shown as the curve of E1 in Fig. 1. The regions over which node 1 is better than node 2,3 and 5.6 are shown in Fig. 1 as E1,E2 and E3,E4 respectively. Thus when the source in the shadowed region in Fig. 1, node 1 is among the two best nodes.

(c) The alternate route will be selected if the expected number of transmissions into and out of the malfunctioning node is 2 or greater. Let the packet dropping probability be p. The last successful transmission will have probability $(1-p)$. Then the expected number of transmissions is

$$1(1-p) + 2p(1-p) + 3p^2(1-p) + \cdots = 2$$

Solving above equality we have $\sum_{i=0}^{\infty}(i+1)p^i(1-p) = 1/(1-p) = 2$ and hence, $p = 0.5$.

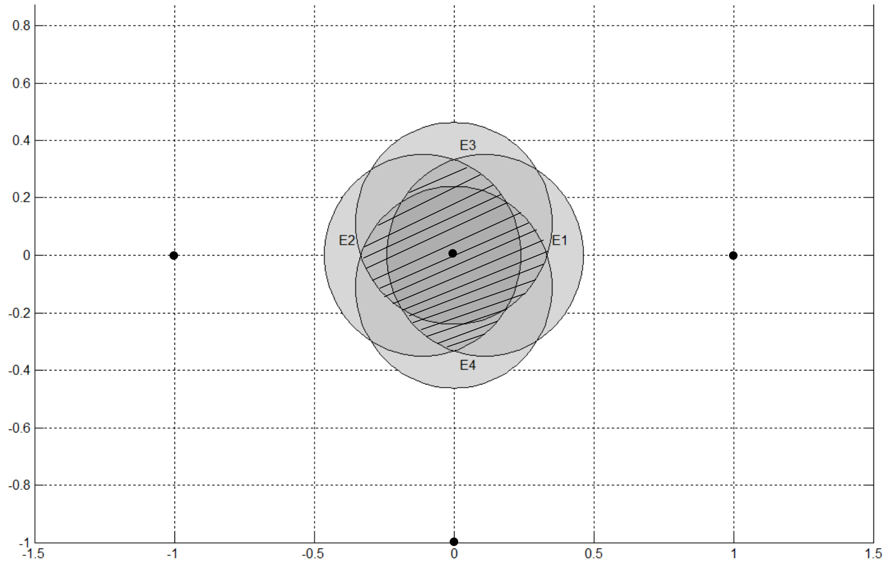

Figure 1: The region over which node 1 is better than others.

(d) The probability of requiring less than or equal to 3 transmissions is

$$(1-p)+p(1-p)+p^2(1-p)=0.99$$

This is a cubic in p and can be solved in any number of ways. A probability of 0.2 is close. Thus the delay requirement leads more quickly to choice of alternative paths in this example.

5. (a) Since $\tau < h$, at most two controllers samples need be applied during the k-th sampling period: $u((k-1)h)$ and $u(kh)$. The dynamical system can be rewritten as

$$\begin{aligned}
\dot{x}(t) &= Ax(t)+Bu(t), \qquad t\in[kh+\tau,(k+1)h+\tau) \\
y(t) &= Cx(t)\,, \\
u(t^+) &= -Kx(t-\tau), \qquad t\in\{kh+\tau, \quad k=0,1,2,\ldots\}
\end{aligned}$$

where $u(t^+)$ is a piecewise continuous and changes values only at $kh+\tau$. By sampling the system with period h, we obtain

$$\begin{aligned}
x((k+1)h) &= \Phi x(kh)+\Gamma_0(\tau)u(kh)+\Gamma_1(\tau)u((k-1)h) \\
y(hk) &= Cx(kh)\,,
\end{aligned}$$

where

$$\begin{aligned}
\Phi &= e^{Ah}=e^{ah}\,, \\
\Gamma_0(\tau) &= \int_0^{h-\tau} e^{As}Bds = \frac{1}{a}\left(e^{a(h-\tau)}-1\right)\,, \\
\Gamma_1(\tau) &= \int_{h-\tau}^{h} e^{As}Bds = \frac{1}{a}\left(e^{ah}-e^{a(h-\tau)}\right)\,.
\end{aligned}$$

given that $A=a, B=1, C=1$.

(b) Let $z(kh) = [x^T(kh), u^T((k-1)h)]^T$ be the augmented state vector, then the augmented closed loop system is

$$z((k+1)h) = \tilde{\Phi} z(kh),$$

where

$$\tilde{\Phi} = \begin{bmatrix} \Phi - \Gamma_0(\tau)K & \Gamma_1(\tau) \\ -K & 0 \end{bmatrix}.$$

Using the results obtained in (a), we can obtain

$$\tilde{\Phi} = \begin{bmatrix} e^{ah} - \frac{1}{a}\left(e^{a(h-\tau)} - 1\right)K & \frac{1}{a}\left(e^{ah} - e^{a(h-\tau)}\right) \\ -K & 0 \end{bmatrix}.$$

(c) The characteristic polynomial of this matrix is

$$\lambda^2 - \left(e^{ah} - \frac{1}{a}\left(e^{a(h-\tau)} - 1\right)\right)K + \frac{K}{a}\left(e^{ah} - e^{a(h-\tau)}\right).$$

Thus when the max $|\lambda| > 1$, the closed loop system becomes unstable.

(d) We use the following result to study the stability of the system:

Theorem 0.1 *Consider the system given in Fig. 2. Suppose that the closed-loop system without packet losses is stable. Then*

- *if the open-loop system is marginally stable, then the system is exponentially stable for all* $0 < r \leq 1$.
- *if the open-loop system is unstable, then the system is exponentially stable for all*

$$\frac{1}{1 - \gamma_1/\gamma_2} < r \leq 1,$$

where $\gamma_1 = \log[\lambda^2_{\max}(\Phi - \Gamma K)]$, $\gamma_2 = \log[\lambda^2_{\max}(\Phi)]$

Here we have

$$\Phi = e^{Ah} = e^{ah},$$

$$\Gamma = \int_0^h e^{As} B ds = \frac{1}{a}\left(e^{ah} - 1\right).$$

Thus, the stability of this system depends on the values of K, h, a. When the conditions are not satisfied, we may choose different K for controller or different sampling time h for the system to make the system stable.