Solutions to Exam in EL2745 Principles of Wireless Sensor Networks, August 20, 2012

1. (a) Node i can win the contention if it picks a slot m while all the other contenders pick slots bigger than m. A slot m is a collision slot if at least two nodes select slot m to transmit and the rest of the nodes select slots bigger than m. Another way of formulating a collision slot is to have all the nodes to pick slots bigger than m. Moreover, there should not exist a single node that picks m and also we should not have a situation where nobody picks slot m.

   (b) Consider a contention round with length M and total number of sensors N. Let x_n be the selected slot of node n. $p_s(m)$ is the probability of having a successful transmission at slot m which happens when a node selects slot m and rest of the nodes select greater slots. P_s is the probability of success over the entire contention round and is obtained by the summation over $p_s(m)$

$$P_s = \sum_{m=1}^{M} p_s(m) = \sum_{m=1}^{M} \sum_{n=1}^{N} \text{Prob}\{x_n = m,\, x_j > m \, \forall j \neq n\}$$

$$= \sum_{m=1}^{M} \binom{N}{1} \frac{1}{M} \left(1 - \frac{m}{M}\right)^{N-1}.$$

   (c) Denote $p_c(m)$ as the probability of collision at slot m then

$$p_c(m) = \text{Prob}\{x_n \geq m,\, \forall n\} \cdot \Big[1 - \text{Prob}\{x_n = m,\, x_j > m \forall j \neq n | x_n \geq m \, \forall n\}$$
$$- \text{Prob}\{x_n \geq m+1 | x_n \geq m,\, \forall n\}\Big]$$
$$= \frac{1}{M^N}\left[(M-m+1)^N - (N+M-m)\cdot(M-m)^{N-1}\right],$$

   which is essentially one minus the probability of having successful or idle slots. Also the probability of having collision after contention round can be formulated in a similar way as success case, i.e.,

$$P_c = \sum_{m=1}^{M} p_c(m) = 1 - P_s.$$

2. (a) See Lecture 5.

(b) The synchronous duty-cycled WSNs can be implemented with the beacon-enabled modality of IEEE 802.15.4. The beacon-enabled modality requires the nodes of the network to be synchronized by the PAN coordinator. The PAN coordinator sends beacon messages to the nodes. Thus, the nodes wake up at the beginning of every super frame to know in which slots they can transmit, and then they go to sleep immediately until they wake up in their time slot during the contention free period.

The asynchronous duty-cycled WSNs can be implemented with the non-beacon enabled modality, or beacon less modality, of IEEE 802.15.4. In the non-beacon enabled modality nodes wake up and transmit exactly as described in the general case. Thus IEEE 802.15.4 is perfectly compatible with asynchronous duty-cycled WSNs.

(c) As the probability to wake up is uniformly distributed in the interval $[0,T]$ the sending node has in average to wait $T/2$ until the receiving node wakes up. Then it takes $L_{ack}/R + t_d$ for the receiving node to transmit the acknowledgement packet.

The probability that the i-th transmission is successful is $P_e^{i-1} \cdot (1-P_e)$. Hence we get for the expected number of transmission attempts N_a:

$$N_\mathrm{a} = \sum_{i=1}^{\infty} i P_e^{i-1} (1-P_e) = \frac{1}{1-P_e}. \tag{1}$$

The transmission for the actual packet takes $L/R + t_d$ and of the acknowledgement packet $L_{ack}/R + t_d$. In the last attempt we don't count the time for acknowledgement package. Hence we get for expected time for the overall transmission:

$$\mathrm{E}\left[t_\mathrm{total}\right] = \frac{T}{2} + t_d + \frac{1}{1-P_e}\left(\frac{L+L_{ack}}{R} + 2t_d\right). \tag{2}$$

3. (a) The expected number of transmissions between i and j is $1/p_{ij}$.

(b) We want to use the "ETX metric" to find the routing tree. Node 1 in figure 1 is the sink. $\mathcal{P}_i$ is the parent of the i-th node. According to Figure 1, nodes update their ETX in the following manner:

$$\text{ETX}[1] = 0$$

node 2:

$$\text{ETX}[2] = \min\{\tfrac{1}{0.5},\tfrac{1}{0.5}+\text{ETX}[7]\} = \min\{2,\infty\} = 2,\ \mathcal{P}_2 = \{1\}$$

node 3:

$$\text{ETX}[3] = \min\{\tfrac{1}{1},\tfrac{1}{0.5}+\text{ETX}[5],\tfrac{1}{0.5}+\text{ETX}[6]\} = \min\{1,\infty,\infty\} = 1,\\ \mathcal{P}_3 = \{1\}$$

node 4:

$$\text{ETX}[4] = \min\{\tfrac{1}{1},\tfrac{1}{0.5}+\text{ETX}[5],\tfrac{1}{0.5}+\text{ETX}[6]\} = \min\{1,\infty,\infty\} = 1,\\ \mathcal{P}_4 = \{1\}$$

node 5:

$$\text{EXT}[5] = \min\{\tfrac{1}{0.5}+\text{ETX}[3],\tfrac{1}{0.5}+\text{ETX}[4],\tfrac{1}{0.5}+\text{ETX}[7]\ \}\\ = \min\{2+1,2+1,\infty\} = 3,\ \mathcal{P}_5 = \{3\}$$

node 6:

$$\text{EXT}[6] = \min\{\tfrac{1}{0.5}+\text{ETX}[3],\tfrac{1}{0.5}+\text{ETX}[4],\tfrac{1}{0.5}+\text{ETX}[7]\ \}\\ = \min\{2+1,2+1,\infty\} = 3,\ \mathcal{P}_6 = \{3\}$$

node 7:

$$\text{EXT}[7] = \min\{\tfrac{1}{0.5}+\text{ETX}[2],\tfrac{1}{0.5}+\text{ETX}[5],\tfrac{1}{0.5}+\text{ETX}[6]\ \}\\ = \min\{2+2,\ 2+3,\ 2+3\} = 4,\ \mathcal{P}_7 = \{2\}$$

In the next iteration the ETX values will remain unchanged and the algorithm converges. The topology is then:

$$\mathcal{P}_{2,3,4,5,6,7} = \{1,1,1,3,3,2\}.$$

(c)

$$[d_{ij}] = \begin{bmatrix} 0 & 2 & 1 & 1 & \infty & \infty & \infty \\ 2 & 0 & \infty & \infty & \infty & \infty & 2 \\ 1 & \infty & 0 & \infty & 2 & 2 & \infty \\ 1 & \infty & \infty & 0 & 2 & 2 & \infty \\ \infty & \infty & 2 & 2 & 0 & \infty & 2 \\ \infty & \infty & 2 & 2 & \infty & 0 & 2 \\ \infty & 2 & \infty & \infty & 2 & 2 & 0 \end{bmatrix} \tag{3}$$

The iterations of the Bellman-Ford algorithm are according to Table 1.

Table 1: Bellman-Ford algorithm steps.

h	D_1^h	D_2^h	D_3^h	D_4^h	D_5^h	D_6^h	D_7^h
1	0	2	1	1	∞	∞	∞
2	0	2	1	1	3	3	4
3	0	2	1	1	3	3	4

4. (a) Denote the unknown position as (x,y). (x,y) may be found by solving the following nonlinear system of equations

$$(x-x_1)^2+(y-y_1)^2=r_1^2=9$$
$$(x-x_2)^2+(y-y_2)^2=r_2^2=16$$
$$(x-x_3)^2+(y-y_3)^2=r_3^2=25$$

The solution is

$$x=\frac{\left[\begin{array}{ccc} x_1^2-r_1^2 & x_2^2-r_2^2 & x_3^2-r_3^2 \end{array}\right]\cdot\left[\begin{array}{c}(y_3-y_2)\\(y_1-y_3)\\(y_2-y_1)\end{array}\right]+(y_3-y_2)(y_1-y_3)(y_2-y_1)}{x_1(y_3-y_2)+x_2(y_1-y_3)+x_3(y_2-y_1)}=0;$$

$$y=\frac{\left[\begin{array}{ccc} y_1^2-r_1^2 & y_2^2-r_2^2 & y_3^2-r_3^2 \end{array}\right]\cdot\left[\begin{array}{c}(x_3-x_2)\\(x_1-x_3)\\(x_2-x_1)\end{array}\right]+(x_3-x_2)(x_1-x_3)(x_2-x_1)}{y_1(x_3-x_2)+y_2(x_1-x_3)+y_3(x_2-x_1)}=0.$$

(b) Notice that while points A and B's measured ranges intersect at point D, point C's measured range cannot go through that point. This discrepancy indicates that there is a measurement error. Since any measurement error or offset has in this case been assumed to affect all measurements, we should look for a single correction factor that would allow all the measurements to intersect at one point. In our example, one discovers that by subtracting 0.5 meter from each measurement the ranges would all intersect at one point. After finding that correction factor, the receiver can then apply the correction to all measurements.

(c) Expanding the equations, we have

$$\begin{cases} (0-x_1)^2+(0-y_1)^2+(0-z_1)^2+2\left[\begin{array}{ccc}(0-x_1) & (0-y_1) & (0-z_1)\end{array}\right]\left[\begin{array}{c}x\\y\\z\end{array}\right]=d_1^2 \\ (0-x_2)^2+(0-y_2)^2+(0-z_2)^2+2\left[\begin{array}{ccc}(0-x_2) & (0-y_2) & (0-z_2)\end{array}\right]\left[\begin{array}{c}x\\y\\z\end{array}\right]=d_2^2 \\ (0-x_3)^2+(0-y_3)^2+(0-z_3)^2+2\left[\begin{array}{ccc}(0-x_3) & (0-y_3) & (0-z_3)\end{array}\right]\left[\begin{array}{c}x\\y\\z\end{array}\right]=d_3^2 \end{cases}$$

or

$$\begin{cases} x_1^2+y_1^2+z_1^2-d_1^2 = 2\begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ x_2^2+y_2^2+z_2^2-d_2^2 = 2\begin{bmatrix} x_2 & y_2 & z_2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ x_3^2+y_3^2+z_3^2-d_3^2 = 2\begin{bmatrix} x_3 & y_3 & z_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \end{cases}$$

which can be reorganized into

$$2\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & y_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_1^2+y_1^2+z_1^2-d_1^2 \\ x_2^2+y_2^2+z_2^2-d_2^2 \\ x_3^2+y_3^2+z_3^2-d_3^2 \end{bmatrix}.$$

5. (a) The are two sources of delay in the network, the sensor to controller τ_{sc} and controller to actuator τ_{ca}. The the control low is fixed. Therefore, these delays can be lumped together for analysis purposed: $\tau = \tau_{sc} + \tau_{ca}$.
Since $\tau < h$, at most two controllers samples need be applied during the k-th sampling period: $u((k-1)h)$ and $u(kh)$. The dynamical system can be rewritten as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad t \in [kh+\tau, (k+1)h+\tau) \\ y(t) &= Cx(t)\,, \\ u(t^+) &= -Kx(t-\tau), \qquad t \in \{kh+\tau, \quad k = 0,1,2,\ldots\} \end{aligned}$$

where $u(t^+)$ is a piecewise continuous and changes values only at $kh+\tau$. By sampling the system with period h, we obtain

$$\begin{aligned} x((k+1h)) &= \Phi x(kh) + \Gamma_0(\tau)u(kh) + \Gamma_1(\tau)u((k-1)h) \\ y(hk) &= Cx(kh)\,, \end{aligned}$$

where

$$\begin{aligned} \Phi &= e^{Ah}\,, \\ \Gamma_0(\tau)u(kh) &= \int_0^{h-\tau} e^{As}Bds\,, \\ \Gamma_1(\tau)u((k-1)h) &= \int_{h-\tau}^{h} e^{As}Bds\,. \end{aligned}$$

(b) Let $z(kh) = [x^T(kh), u^T((k-1)h)]^T$ be the augmented state vector, then the augmented closed loop system is

$$z((k+1)h) = \tilde{\Phi} z(kh)\,,$$

where

$$\tilde{\Phi} = \begin{bmatrix} \Phi - \Gamma_0(\tau)K & \Gamma_1(\tau) \\ -K & 0 \end{bmatrix}.$$

(c) Given that $A = 0$ and $B = I$, we have

$$\tilde{\Phi} = \begin{bmatrix} 1 - hK + \tau K & \tau \\ -K & 0 \end{bmatrix}.$$

The characteristic polynomial of this matrix is

$$\lambda^2 - (1 - Kh + \tau K)\lambda + \tau K\,.$$

By recalling that $h = 1/K$, we define $y = \tau/h$, so it follows that the characteristic polynomial is

$$\lambda^2 - y\lambda + y\,.$$

The solutions λ_1 and λ_2 to such an equation are

$$\lambda_1 = \frac{y}{2} + j\frac{1}{\sqrt{4y - y^2}},$$
$$\lambda_2 = \frac{y}{2} - j\frac{1}{\sqrt{4y - y^2}}.$$

Since $|\lambda| < 1$, there is no other constraint for λ