EL2520
Control Theory and Practice

Lecture 2: The closed-loop system

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Goals

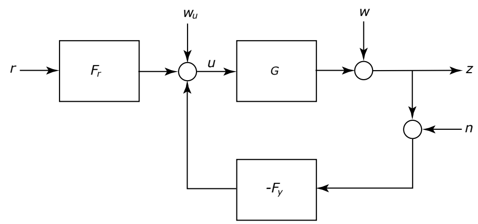
After this lecture, you should:

- Know that the closed-loop is characterized by 6 transfer functions
  - Dangerous to design for only one
  - Cancellations and the concept of internal stability
- Determine, analyze and design desired sensitivity functions
  - Sensitivity function for disturbance rejection
  - Complementary sensitivity function for robust stability
- Understand limitations and conflicts, relation to stability margins

Material: course book Chapter 6.

Contents

The closed-loop system



Controller: feedback F_y, feedforward F_r
Disturbances: w, w_u drive system from desired state
Measurement noise: corrupts information about z

Aim: find controller such that z follows r, with limited use of u

The design problem

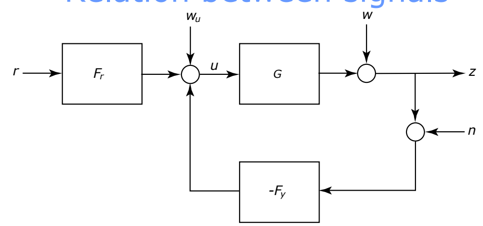
Design problem: find a controller that

a) Reduces the effect of load disturbances
b) Does not inject too much measurement noise into the system
c) Makes the closed loop insensitive to process variations
d) Makes the output follow command signals

Often convenient with two-degree of freedom controller
(separate transmission from y→u and from r→u)

Use feedback to deal with a,b,c; use feedforward to deal with d.

Relation between signals



$$z = w + G(\,w_u + F_r r - F_y(z+n)\,) \Rightarrow$$

$$z = \underbrace{\frac{1}{1+GF_y}}_{S} w + \underbrace{\frac{G}{1+GF_y}}_{SG} w_u + \underbrace{\frac{GF_r}{1+GF_y}}_{G_c} r - \underbrace{\frac{GF_y}{1+GF_y}}_{T} n$$

Similarly, we find

$$u = SF_r r - SF_y(w+n) + Sw_u$$

Closed-loop for SISO G(s) characterized by *six* transfer functions

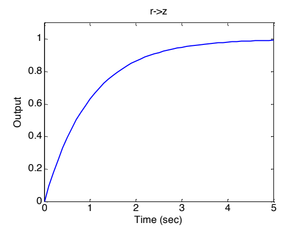
Transfer functions and observations

$S = \dfrac{1}{1+GF_y}$	$(w \to z,\ w_u \to u)$	sensitivity function
$T = \dfrac{GF_y}{1+GF_y}$	$(n \to z)$	complementary sensitivity
$G_c = \dfrac{GF_r}{1+GF_y}$	$(r \to z)$	closed loop system
$SG = \dfrac{G}{1+GF_y}$	$(w_u \to z)$	
$SF_y = \dfrac{F_y}{1+GF_y}$	$(n \to u)$	
$SF_r = \dfrac{F_r}{1+GF_y}$	$(r \to u)$	

Observation: need to look at all! Many tradeoffs (e.g. S+T=1)

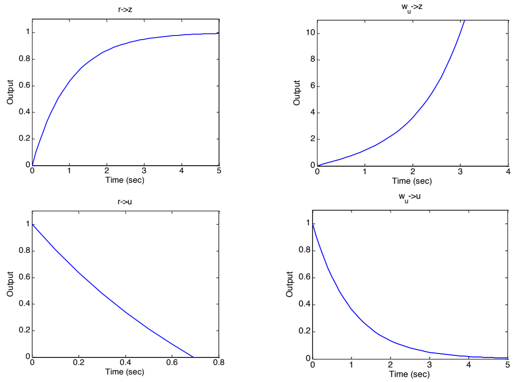
A warning!

Individual time responses may look good



but you need to verify that *all* transfer functions are as desired!

Four responses



What's going on?

Process: $G = \dfrac{1}{s-1}$

Controller: $F_y = F_r = \dfrac{s-1}{s}$

Transfer functions:

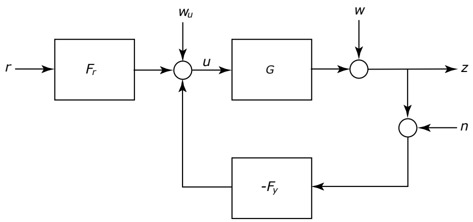
$$T = \frac{1/s}{1+1/s} = \frac{1}{s+1} \qquad \text{(stable)}$$

$$SG = \frac{s}{(s+1)(s-1)} \qquad \text{(unstable!)}$$

$$SF_y = \frac{(s-1)}{(s+1)} \qquad \text{(stable)}$$

$$S = \frac{s}{(s+1)} \qquad \text{(stable)}$$

Internal stability



Definition. The closed loop system above is *internally stable* if it is input-output stable from r, w_u, w, n to all outputs u, z, y.

Theorem. If G is SISO, the closed-loop system is internally stable stable if and only if $S,\ SG,\ SF_y,\ F_r$ are stable

Sensitivity functions

Sensitivity and complementary sensitivity are particularly important:

- S determines suppression of load disturbances,
- T determines robustness to noise and unmodelled dynamics

Both connected to classical stability margins (gain, phase margin)

First trade-off: **S+T=1** - can't make both zero at the same time.

Disturbance rejection

The transfer function from w to z in open loop is

$$G^{\text{ol}}_{w\to z} = 1$$

while the closed-loop counter-part is

$$G^{\text{cl}}_{w\to z} = \frac{1}{1+GF_y}$$

Thus

$$\frac{G^{\text{cl}}_{w\to z}}{G^{\text{ol}}_{w\to z}} = \frac{1}{1+GF_y}$$

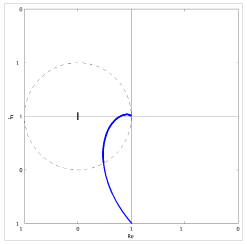
S quantifies the disturbance attenuation due to closed-loop control.

Disturbances at frequencies with $S(i\omega) \geq 1$ amplified by feedback!

Nyquist curve interpretation

$|S(i\omega)| = |L(i\omega)+1|^{-1}$ is inverse distance from Nyquist curve to -1

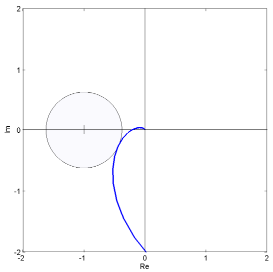
Disturbance attenuation at frequencies where Nyquist curve is inside unit circle centered at the -1 point.



Observation: can't avoid circle if pole excess $\geq$ 2, must amplify disturbances *at some* frequencies (more next lecture!)

Maximum sensitivity and M_s-circles

Specification $|S(i\omega)| \leq M_s$: loop gain outside circle with radius M_s^{-1}



Reasonable values: $1.2 \leq M_s \leq 2$
(picture shows M_s=2)

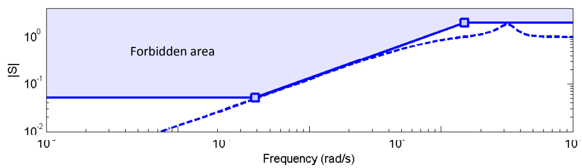
Sensitivity shaping

Observations:

- Can't attenuate disturbances at all frequencies (if pole excess $\geq$ 2)
- Need to limit $|S(i\omega)|$ at frequencies with significant disturbances

Reasonable design specification

$$|S(i\omega)| \leq \left|W_S^{-1}(i\omega)\right| \;\forall\omega$$



Sensitivity to uncertainties

Sensitivity of what, and to what?

- Sensitivity of closed-loop transfer function to model uncertainties

The response of z to r (assuming no disturbances) is

$$z = G_{\mathrm{cl}} r = \frac{GF_r}{1 + GF_y} r$$

If there is a model error, so that the true system is

$$\tilde{G} = (1 + \Delta_G) G$$

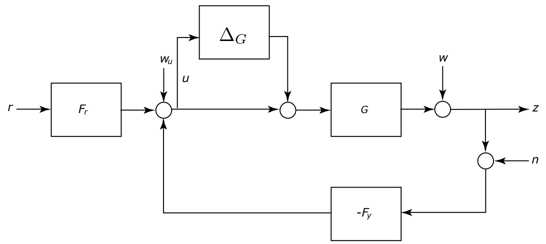
then the true response to r is

$$\tilde{z} = (1 + \tilde{S}\Delta_G) z$$

Robust stability

Uncertainties also affect stability of closed-loop system.

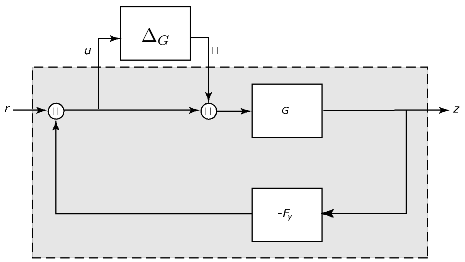
Assume true system is given by $\tilde{G} = (1 + \Delta_G) G$



What linear Δ_G can be tolerated without jeopardizing stability?

Robust stability via small gain theorem

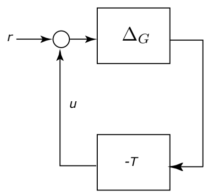
Assume all exogenous inputs (r, w, w_u, n) to be zero, re-write



Note that

$$u = -\frac{GF_y}{1 + GF_y} = -T$$

Robust stability via small gain theorem



Assume Δ_G stable and nominal-system T internally stable. If

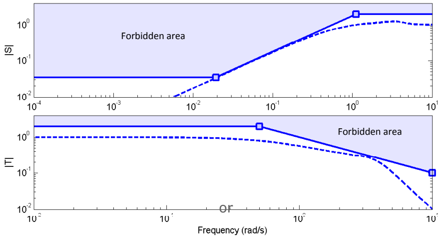
$$\|T\Delta_G\|_\infty \leq 1$$

then the above system (and, hence, the original system) is input-output stable.

Proof. Small-gain theorem

Sensitivity shaping

Natural design criterion: make sure that both the sensitivity S and the complementary sensitivity T avoid "forbidden areas"



or

$$|S(i\omega)| \le |W_S^{-1}(i\omega)| \qquad \|SW_S\|_\infty \le 1$$
$$|T(i\omega)| \le |W_T^{-1}(i\omega)| \qquad \|TW_T\|_\infty \le 1$$

Extension: shaping frequecy responses

Can shape all relevant transfer functions (in "the gang of six")

$$\|SW_S\|_\infty \le 1$$
$$\|TW_T\|_\infty \le 1$$
$$\vdots$$
$$\|SF_rW_{SF_r}\|_\infty \le 1$$

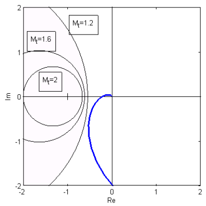
This is the topic of Computer Exercise 1b!

Complementary sensitivity in Nyquist curve

Constraint on complementary sensitivity

$$\|T\|_\infty \le M_t$$

also yields circles that should be avoided by the Nyquist curve.

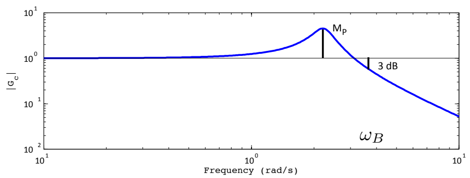


Circles centered at $(-M_t^2/(M_t^2-1),\ 0)$ with radius $M_t/(M_t^2-1)$

Closed loop transfer function and tracking

Reference following determined by closed-loop transfer function

$$y = G_c r = \frac{GF_r}{1+GF_y} r = GSF_r r$$



Design criterion: choose F_r so that M_p and ω_B equal desired values

Note: potential conflict with S, control signal limitations

Steady-state errors

Step in reference signal

$$e_0 = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s(1 - G_c(s))\frac{1}{s}$$

If $F_r = F_y$, then $1 - G_c(s) = S(s)$ and $e_0 = 0$ requires $S(0) = 0$

True if $\lim_{\omega\to 0} G(\omega)F_y(\omega) = \infty$, i.e. if integrator in F_y or G.

Tracking a ramp signal requires two integrators, etc.

Perfect suppression of disturbances treated analogously.

Summary

- Closed-loop system characterized by 6 transfer functions
  - Need to consider all!
- Sensitivity and complementary especially important
  - S: disturbance attenuation, "performance sensitivity"
  - T: noise attenuation, robust stability
  - Close relationship with classical stability margins
- Control system design via "sensitivity shaping"
- Conflicts and limitations
  - S+T=1
  - $|S(i\omega)| \geq 1$ for some ω (disturbance amplification!)
  - Much more next lecture!