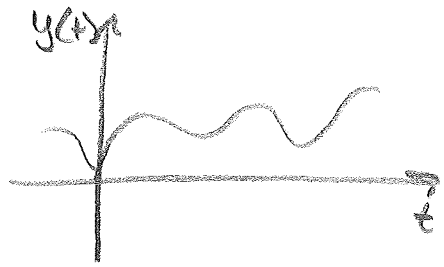
Theory:

Norms: Good for comparing signals & systems.

Signal norms:

$y(t)$ - a signal



How "big" is this signal?

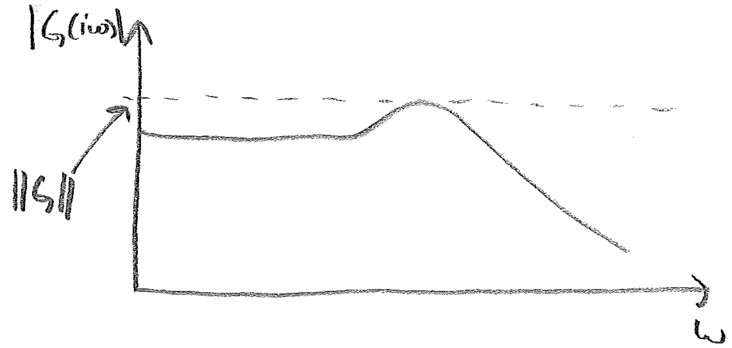
$$\|y\|_2 = \left(\int_{-\infty}^{\infty} |y(t)|^2 dt\right)^{1/2} \quad \text{(energy)}$$

$$\|y\|_\infty = \sup_t |y(t)| \quad \text{(peak value)}$$

System norm: How much "bigger" can the system make a signal

Linear systems: H_∞-norm

$$\|G\| := \sup_\omega |G(i\omega)| = \| |G(i\omega)| \|_\infty$$

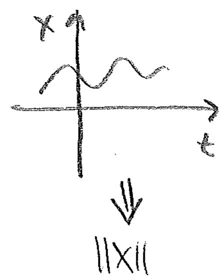


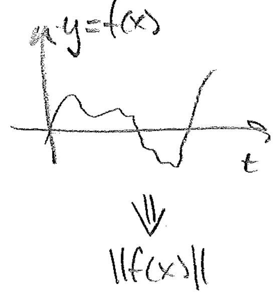
Peak in bode-diagram.

Static nonlinearity: x & y are signals

$$y = f(x)$$

$$\|f\| := \sup_x \frac{\|f(x)\|_2}{\|x\|_2}$$



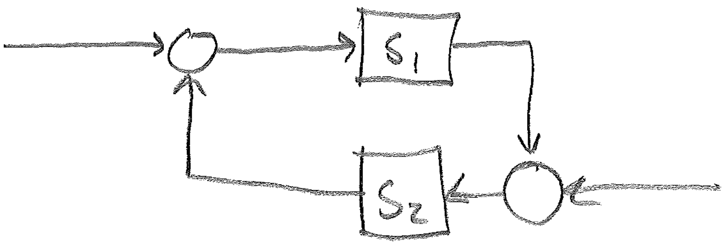


Useful inequalities

$\|A+B\| \leq \|A\| + \|B\|$ (triangle ineq.) Valid for all norms

$\|A \cdot B\| \leq \|A\| \cdot \|B\|$ (sub-multiplicative) Valid for some norms, e.g. H_∞.

Small gain theorem:

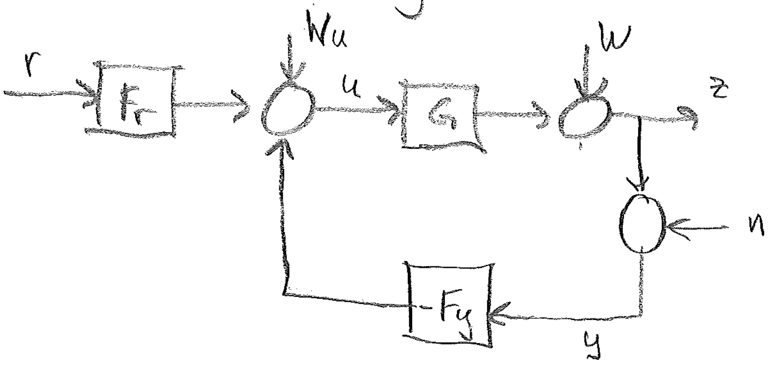


If S_1 & S_2 are stable and

Linear case	nonlinear case
$\|S_1 S_2\| < 1$	$\|S_1\| \cdot \|S_2\| < 1$

then the closed loop is stable.

Internal stability



A system is internally stable if all signals remain bounded after injecting a bounded disturbance anywhere in the system.

$\Rightarrow$ Need to look at "all" transfer functions from $r, W_u, W, n \longrightarrow z, y, u$

Theorem: (SISO)

Internally stable if and only if

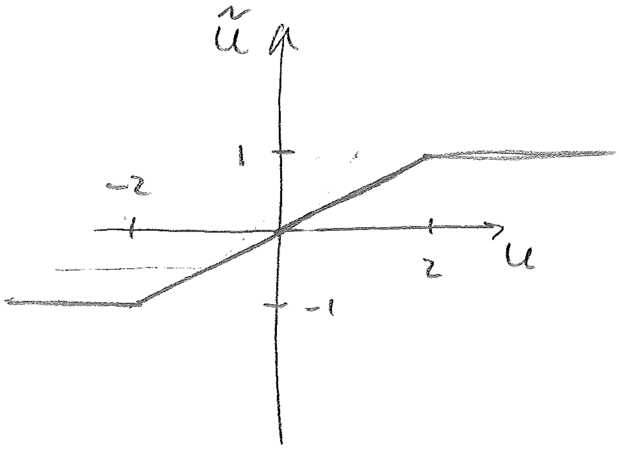
$$S = \frac{1}{1+F_y G}, \quad SG = \frac{G}{1+F_y G}, \quad SF_y = \frac{F_y}{1+F_y G}, \quad F_r$$

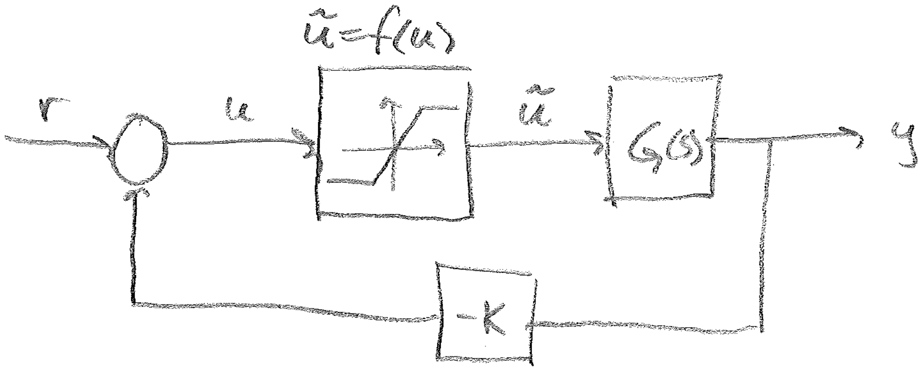
are all stable.

1,3

$$G(s) = \frac{2}{s^2+2s+2}$$

$$\tilde{u} = \begin{cases} 1 & u > 2 \\ \frac{1}{2}u & |u| < 2 \\ -1 & u < -2 \end{cases} \Rightarrow$$





For which values of K are the system stable according to the small gain theorem?

Since we have a nonlinearity we need to use the "non-linear version" of the small-gain theorem

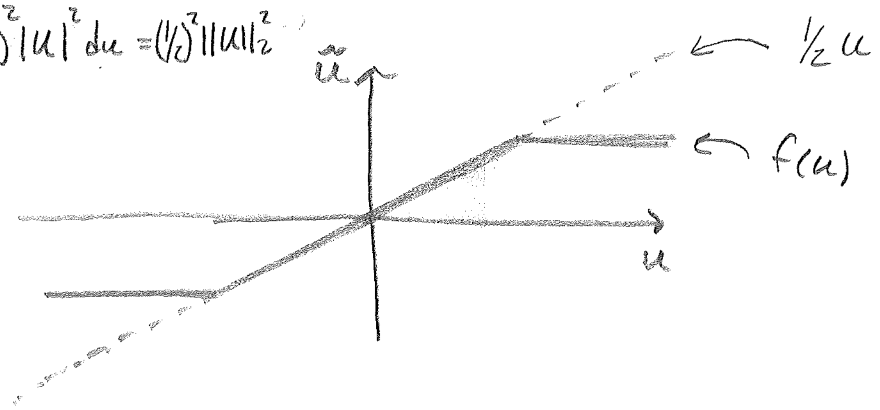
$\|f\| \cdot \|G\| \cdot \|K\| < 1 \Rightarrow$ stable if all subsystems stable.

Is f input-output stable? Yes, since the output is bounded $|f(u)| < 1 \ \forall u$.

$\|f\|$: We note that $|\hat{u}| = |f(u)| \leq \frac{1}{2}|u| \Rightarrow$

$$\|\hat{u}\|_2^2 = \int_{-\infty}^{\infty} |\hat{u}|^2 du \leq \int_{-\infty}^{\infty} (1/2)^2 |u|^2 du = (1/2)^2 \|u\|_2^2$$

$\Rightarrow \|f\| \leq 1/2$



$\|G\|$: From the definition we have

$$\|G\| = \sup_{\omega} |G(i\omega)|$$

$$|G(i\omega)| = \frac{2}{|i^2\omega^2 + 2i\omega + 2|} = \frac{2}{\sqrt{4\omega^2 + (2-\omega^2)^2}}$$

Maximizing $|G(i\omega)|$ is the same as minimizing $4\omega^2 + (2-\omega^2)^2$

$$\frac{d}{d\omega} 4\omega^2 + (2-\omega^2)^2 = 8\omega - 4\omega(2-\omega^2) = 4\omega^3$$

We see that this is only zero at $\omega = 0 \Rightarrow$

$$\|G\| = |G(i0)| = 1$$

Alternative: If we rewrite the system as

$$G(s) = \frac{\omega_0^2}{s^2 + 2\omega_0 \zeta \omega + \omega_0^2} \Rightarrow \zeta = \frac{1}{\sqrt{2}} \Rightarrow$$ system is well damped seond order system.

$$\Rightarrow \|G\| = \sup_{\omega} |G(i\omega)| = G(0) = 1$$

$\|{-K}\|$:

The gain of a constant is the constant

$\|{-K}\| = |K|$.

For stability we then get by the S.G.T.

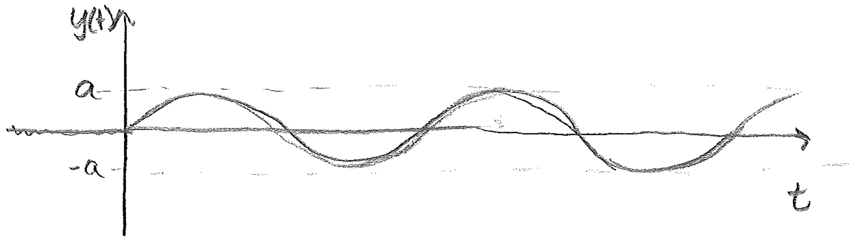
$\|f\| \cdot \|G\| \cdot \|{-K}\| = \frac{1}{2} \cdot 1 \cdot |K| < 1 \Rightarrow \underline{\underline{|K| < 2}}$

Exercise 1,4

Calculate $\|\cdot\|_\infty$ & $\|\cdot\|_2$ for the signals

a, $y(t) = \begin{cases} a\sin(t) & t > 0 \\ 0 & t \leq 0 \end{cases}$

$$\|y(t)\|_\infty = \sup_t |y(t)| = \sup_t |a\sin(t)| = |a| \underbrace{\sup_t |\sin(t)|}_{1} = |a|$$



$$\|y(t)\|_2 = \sqrt{\int_{-\infty}^{\infty} y(t)^2\, dt} = \sqrt{\int_0^\infty a^2 \sin^2 t\, dt} =$$

$$= |a| \sqrt{\int_0^\infty \sin^2 t\, dt} = / \sin^2 t = 1/2 - 1/2 \cos 2t /$$

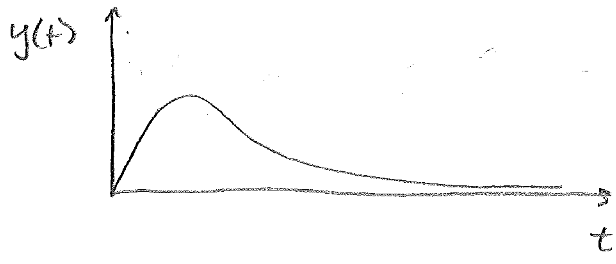
$$= |a| \sqrt{\int_0^\infty 1/2\, dt - \underbrace{\int_0^\infty 1/2 \cos 2t\, dt}_{< 1 \ \forall t > 0}} > |a| \sqrt{\infty - 1} = \infty$$

b, $y(t) = \begin{cases} 1/t & t > 1 \\ 0 & t \leq 1 \end{cases}$

$$\|y(t)\|_\infty = \sup_{t>1} |1/t| = 1$$

$$\|y(t)\|_2 = \sqrt{\int_1^\infty 1/t^2\, dt} = \sqrt{\left[\frac{-1}{t}\right]_1^\infty} = \sqrt{0-(-1)} = 1$$

c)
$$y(t) = \begin{cases} e^{-t}(1-e^{-t}) & t > 0 \\ 0 & t \leq 0 \end{cases}$$



$$\|y(t)\|_\infty = \sup_{t>0} e^{-t}(1-e^{-t}) = \sup_{t>0} e^{-t} - e^{-2t}$$

We take the derivative and equate to zero in order to find the max.

$$\frac{dy}{dt} = -e^{-t} + 2e^{-2t} = 0 \iff \underbrace{e^{-t}}_{>0\,\forall t}(2e^{-t}-1) = 0$$

$$2e^{-t} = 1 \iff \log(2e^{-t}) = 0$$
$$\iff \log 2 + \log(e^{-t}) = 0$$
$$\iff t = \log 2$$

So we get the maximum at $t = \log(2)$

$$y(\log(1/2)) = e^{-\log(2)} - e^{-2\log(2)} = e^{\log(1/2)} - e^{\log(1/4)} =$$
$$= 1/2 - 1/4 = 1/4$$

$$\|y(t)\|_2 = \sqrt{\int_0^\infty (e^{-t} - e^{-2t})^2 \, dt} =$$
$$= \sqrt{\int_0^\infty (e^{-2t} - 2e^{-3t} + e^{-4t}) dt} = \sqrt{\left[\frac{-1}{2}e^{-2t} + \frac{2}{3}e^{-3t} - \frac{1}{4}e^{-4t}\right]_0^\infty} =$$
$$= \sqrt{\frac{1}{2} - \frac{2}{3} + \frac{1}{4}} = \frac{1}{\sqrt{12}}$$

1.5

$$\|G\| = \| \, |G(i\omega)| \, \|_\infty = \sup_\omega |G(i\omega)|$$

$$|G(i\omega)| = \left| \frac{\overset{\text{constant}}{\omega_0^2}}{i^2\omega^2 + 2\zeta\omega_0 i\omega + \omega_0^2} \right| = \frac{\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\zeta^2\omega_0^2\omega^2}}$$

Maximizing $|G(i\omega)|$ is the same as minimizing $(\omega_0^2\omega^2)^2 + 4\zeta^2\omega_0^2\omega^2$

$$\frac{d}{d\omega}(\omega_0^2-\omega^2)^2 + 4\zeta^2\omega_0^2\omega^2 = 2(\omega_0^2-\omega^2)\cdot -2\omega + 8\zeta^2\omega_0^2\omega =$$

$$= 4\omega(2\zeta^2\omega_0^2 - \omega_0^2 + \omega^2) =$$

$$= 4\omega(\omega_0^2(2\zeta^2-1) + \omega^2) = 0$$

We get the solutions

$\omega = 0$

$\omega = (\pm)\,\omega_0\sqrt{1-2\zeta^2}$

We require a real positive solution so we need $\zeta < \frac{1}{\sqrt{2}}$

We get two cases:

(I) $0 < \zeta < \frac{1}{\sqrt{2}}$, a porly damped system

$$\|G\| = |G(i\omega_0\sqrt{1-2\zeta^2})| = \frac{\omega_0^2}{\sqrt{(\omega_0^2 - \omega_0^2(1-\zeta^2))^2 + 4\zeta^2\omega_0^4(1-\zeta^2)}} =$$

$$= \frac{\omega_0^2}{\sqrt{4\omega_0^4\zeta^2(1-\zeta^2)}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} > 1$$

(II) $\zeta > \frac{1}{\sqrt{2}} \Rightarrow \|G\| = |G(i0)| = \frac{\omega_0^2}{0+0+\omega_0^2} = 1$

1.6

Small gain theorem:

Both systems need to be stable so we must have $a>0$.

By the the S.G.T. we need

$\| K \cdot \frac{a}{s+a} \| < 1$ for stability $\Rightarrow$

$$\sup_{\omega} \frac{Ka}{\sqrt{\omega^2+a^2}} < 1$$

To maximize this we want ω^2+a^2 as small as possible $\Rightarrow$ $\omega=0$

We get $\| K \cdot \frac{a}{s+a} \| = \left| \frac{Ka}{a} \right| = |K|$

Hence $|K|<1$ & $a>0$ for stability

Poles of Closed-loop system:

The closed loop-system becomes

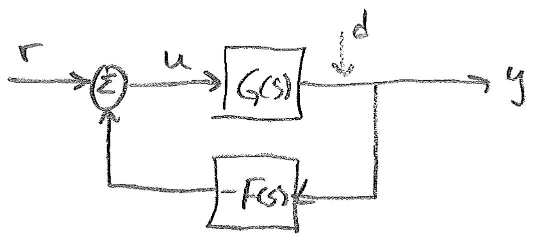
$$G_c(s) = \frac{K \frac{a}{s+a}}{1 - K\frac{a}{s+a}} = \frac{Ka}{s+a(1-K)}$$

Utvecklat & stabilitet

Case 1: $a>0$ We see that we need $K<1$ for stability (negative pole)

Case 2: $a<0$ We need $K>1$ for stability.

6.2



$$G(s) = \frac{s-1}{s+1}, \quad F(s) = \frac{s+2}{s-1}$$

Compute S, T & G_c

G_c: $\quad Y(s) = G_c(s) R(s)$

$$Y(s) = G(s) R(s) - G(s) F(s) Y(s)$$

$$\Rightarrow G_c(s) = \frac{G(s)}{1 + G(s)F(s)} = \frac{\frac{s-1}{s+1}}{1 + \frac{s+2}{s-1}\frac{s-1}{s+1}} = \frac{s-1}{2s+3}$$

S: $\quad L(s) = F(s) G(s)$

$$S(s) = \frac{1}{1 + L(s)} = \frac{1}{1 + \frac{s+2}{s-1}\frac{s-1}{s+1}} = \frac{s+1}{2s+3}$$

T: $\quad T = 1 - S = \frac{2s+3}{2s+3} - \frac{s+1}{2s+3} = \frac{s+2}{2s+3}$

All of these are stable. However, consider the effect of a disturbance on the control signal.

G_{du}: $\quad U(s) = -F(s) G(s) U(s) - F(s) D(s)$

$$\Rightarrow$$

$$G_{du}(s) = \frac{-F(s)}{1 + F(s)G(s)} = -F(s) S(s) = \frac{s+2}{s-1}\frac{s+1}{2s+3}$$

Hence, the control is unstable and grows unbounded for small disturbances.

We are not internally stable.