EL2520
Control Theory and Practice

Lecture 3: Robustness

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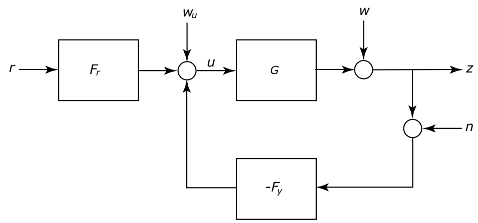
So far…



- Signal norms, system gains and the small gain theorem
- The closed-loop system and the design problem
  - Characterized by six transfer functions: need to look at all!
  - Internal stability: stability from all inputs to all outputs (sufficient to check that F_r, S, SG and SF_y are all stable)
  - Sensitivity function (suppression of load disturbances) and Complementary sensitivity (noise, robust stability)

Goals

After this lecture, you should

- Understand the concepts of robust stability and robust performance

- Be able to derive multiplicative uncertainty models
  - from parametric uncertainties (e.g. of process pole/zero locations)
  - from frequency responses of multiple plants

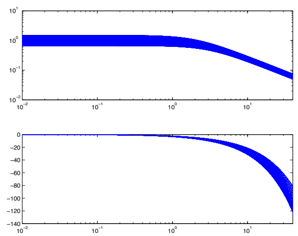
- Analyze robust stability using the small-gain theorem
  - "pull out" uncertainty and re-write system on standard form
  - assess robust stability in Bode and Nyquist diagrams

Motivating example

Assume that you want to control a system on the form

$$G_p(s) = \frac{k}{1+s\tau}e^{-s\theta}$$

but the values of $k,\ \tau,\ \theta$ are unknown. You only know that $k,\ \tau,\ \theta \in [2,3]$



How can we design a controller that is guaranteed to work for all G_p?

Robustness

Robustness=Insensitivity to model errors
(differences between modelled and actual system behavior)

To reason about uncertainty we need to model it!

- The *uncertainty set:* defines a family of possible models (quantifies how much we do not know about the system)

Would like to establish

- *Robust stability* (stability of all plants in uncertainty set)
- *Robust performance* (meet specs for all plants in uncertainty set)

Classes of uncertainty

Parametric uncertainty:

- Model structure known, but some parameters are uncertain

Dynamic uncertainty:

- Some (often high frequency) dynamics is missing, either by lack of understanding or in order to get a simpler model

Often, we have a combination of the two.

- Convenient to represent in “lumped” form

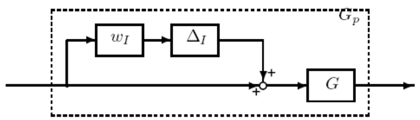
Multiplicative uncertainty

Multiplicative uncertainty

$$\Pi_I = \{G_p(s) = G(s)(1 + W_I(s)\Delta_I(s)) \mid \|\Delta_I\|_\infty \leq 1\}$$

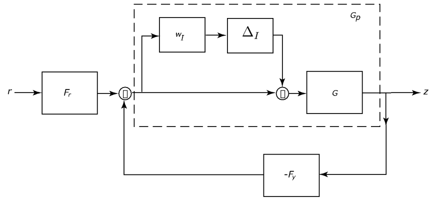
Here,

- Π_I is a *family* of possible behaviours of the physical plant
- Δ is *any* stable transfer function with gain less than one



Robust stability: closed-loop stability for all $G_p \in \Pi_I$

Robust stability w. multiplicative uncertainty



Small-gain theorem→ interconnection stable if
(a) nominal closed-loop system is internally stable and W_I stable, and
(b) $\|W_I T\|_\infty \leq 1$

To ensure robust stability:

- first write uncertain system on standard form (find G(s), W_I(s))
- make sure that (a) and (b) are satisfied

Example: uncertain gain

Consider the set of possible plants

$$G_p(s) = kG_0(s), \quad k_{\min} \le k \le k_{\max}$$

Any feasible k can be written as $k = \overline{k} + r_k\Delta$ for some $|\Delta| \le 1$ and

$$\overline{k} = \frac{k_{\min}+k_{\max}}{2}, \quad r_k = \frac{k_{\max}-k_{\min}}{2}$$

Hence, we can re-write the uncertainty in standard form

$$\Pi_I = \left\{ G_p(s) = \underbrace{\overline{k}G_0(s)}_{G(s)} \Big(1 + \underbrace{\frac{r_k}{\overline{k}}}_{W_I(s)} \Delta\Big) \mid |\Delta| \le 1 \right\}$$

Note: here it is enough to let Δ be real (in standard form Δ is complex)

Example: uncertain zero location

Consider the set of possible plants

$$G_p(s) = (1 + s\tau)G_0(s), \quad \tau_{\min} \le \tau \le \tau_{\max}$$

Can be put into standard form via

$$\begin{aligned} \overline{\tau} &= (\tau_{\min} + \tau_{\max})/2 \\ r_\tau &= (\tau_{\max} - \tau_{\min})/2 \\ G(s) &= (1 + s\overline{\tau})G_0(s) \\ W_I(s) &= \frac{r_\tau s}{1 + \overline{\tau}s} \end{aligned}$$

Note: W_I is now frequency dependent, Δ is still real

Alternative approach to obtain weight

Note that multiplicative uncertainty class

$$\Pi_I = \{G_p(s) = G(s)(1 + W_I(s)\Delta_I(s)) \mid \|\Delta_I\|_\infty \le 1\}$$

can be re-written as

$$\Pi_I = \{G_p(s) \mid \|W_I(s)^{-1}G(s)^{-1}(G_p(s) - G(s))\|_\infty \le 1\}$$

Thus, the uncertainty about the system captured by W_I if

$$|W_I(i\omega)| \ge \left|\frac{G_p(i\omega) - G(i\omega)}{G(i\omega)}\right| \qquad \forall\, G_p \in \Pi_I,\ \forall\, \omega$$

Note: RHS can be interpreted as relative error of nominal model G.

Motivating example cont'd

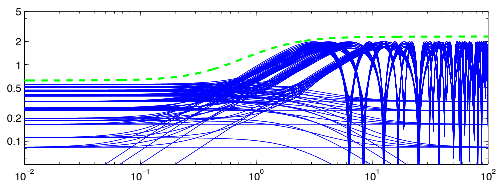
Consider the uncertain system

$$G_p(s) = \frac{k}{\tau s + 1}e^{-\theta s}, \quad k, \theta, \tau \in [2, 3]$$

with nominal plant

$$G(s) = \frac{\overline{k}}{\overline{\tau}s + 1}$$

Sample relative errors (full lines) and corresponding W_I (dashed)



Motivating example cont'd

Assume that we want to control the system using a PI controller

$$C(s) = K_p\left(1 + \frac{1}{sT_i}\right)$$

We tune the gains using the lambda-tuning method

$$K_p = \frac{\bar{\tau}}{\bar{k}(\bar{\theta}+\lambda)}, \quad T_i = \bar{\tau}$$

where lambda is a tuning parameter.

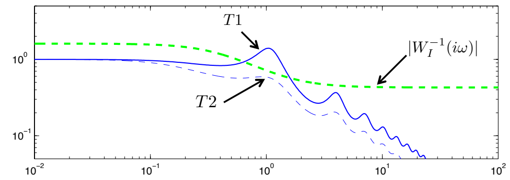
Can we find lambda so that the PI controller guarantees robust stability?

Motivating example cont'd

Robust stability condition $\|W_I T\|_\infty \leq 1$ holds if $|T(i\omega)| \leq |W_I^{-1}(i\omega)| \quad \forall\omega$

Hence, we can validate robust stability in the bode diagram of T.

For two values of lambda, we obtain two complementary sensitivities

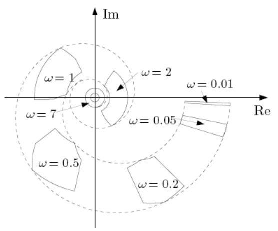


First setting (T1) is *not* robustly stable, second setting (T2) is.

Robust stability in the Nyquist curve

Uncertain system:

- G(iω) takes one of several possible values at each frequency → a family of Nyquist curves



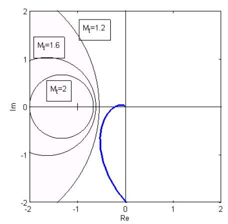
- Robust stability if uncertainty regions do not encircle -1 point

Complementary sensitivity in Nyquist

Constraint on complementary sensitivity

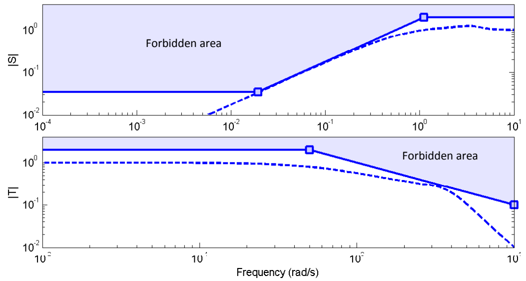
$$\|T(i\omega)\|_\infty \leq M_t$$

also yields circles that should be avoided by the Nyquist curve.



Circles centered at $(-M_t^2/(M_t^2-1),\ 0)$ with radius $M_t/(M_t^2-1)$

Frequency domain specifications



$$|S(i\omega)| \leq |W_S^{-1}(i\omega)|$$
$$|T(i\omega)| \leq |W_T^{-1}(i\omega)|$$

Can we choose weights w_S, w_T ("forbidden areas") freely?
- No, there are many constraints and limitations!

Extension: shaping the gang of six

Can shape all relevant transfer functions (in "the gang of six")

$$\|W_S S\|_\infty \leq 1$$
$$\|W_T T\|_\infty \leq 1$$
$$\vdots$$
$$\|W_{SF_r} SF_r\|_\infty \leq 1$$

This is the topic of Computer Exercise 1b!

Robust performance

Nominal performance specified in terms of sensitivity function

$$|W_P S| \leq 1 \quad \forall \omega$$

Robust performance

$$|W_P S_p| \leq 1 \quad \text{for all } \omega \text{ and all } S_p$$

Since

$$W_P S_p = W_P \frac{1}{1+L_p} = \frac{W_P}{1+L+W_I \Delta L}$$

Worst-case Δ is such that 1+L and $w_I\Delta$ L point in opposite directions

$$|W_P S_p| \leq \frac{|W_P|}{|1+L| - |W_I L|} = \frac{|W_P S|}{1-|W_I T|} \quad \forall \omega$$

Robust performance cont' d

Robust performance

$$|W_P S_p| = \frac{|W_P S|}{1-|W_I T|} \leq 1$$

Can be expressed as

$$|W_P S| + |W_I T| \leq 1 \quad \forall \omega$$

Sometimes approximated by the *mixed* sensitivity constraint

$$\left\| \begin{pmatrix} W_P S \\ W_I T \end{pmatrix} \right\|_\infty \leq 1$$

Robust stability and performance

In summary

nominal performance	$\lvert W_P S \rvert \leq 1 \quad \forall \omega$
robust stability	$\lvert W_I T \rvert \leq 1 \quad \forall \omega$
robust performance	$\lvert W_P S \rvert + \lvert W_I T \rvert \leq 1 \quad \forall \omega$

Note that nominal performance and robust stability implies

$$\lvert W_P S \rvert + \lvert W_I T \rvert \leq 2 \quad \forall \omega$$

(i.e. robust stability cannot be “too bad”).

Only holds in SISO case.

Summary

Robustness
- Insensitivity to model errors

Can guarantee robustness if we model (or bound) uncertainty
- General tool: small gain theorem
- Sometimes need to ”pull out” uncertainty by hand
- Sometimes, can fall back onto standard forms (e.g. multiplicative input uncertainty)

Robustness typically introduces new constraints on T

Robust performance: acceptable S, despite uncertainties.