So far...

- Signal norms, system gains and the small gain theorem
- The closed-loop system and the design problem
  - Characterized by six transfer functions: need to look at all!
  - Internal stability: stability from all inputs to all outputs
    (sufficient to check that $F_r$, $S$, $SG$ and $SF$ are all stable)
  - Sensitivity function (suppression of load disturbances) and Complementary sensitivity (robust stability)

Today’s lecture

- Fundamental limitations in control systems design
  - $S+T=1$ (both can’t be small at the same time)
  - Can’t attenuate disturbances at all frequencies
- Limiting factors:
  - Unstable poles
  - Non-minimum phase zeros
  - Time delays
  - Control authority (exercises; final part of course)
- Reasonable specifications, and rules-of-thumb!

Course book: Chapter 7.
Reasonable specifications

\[ \|SW_S\|_\infty \leq 1 \quad \text{implied by } |S(i\omega)| \leq |W_S^{-1}(i\omega)| \]

\[ \|TW_T\|_\infty \leq 1 \quad \text{implied by } |T(i\omega)| \leq |W_T^{-1}(i\omega)| \]

Specifications in terms of \( W_S \)

\[ W_S^{-1} = M_S \frac{s}{s + \omega_{BS} M_S} \]

Interpolation constraints

**Fact:** Assume that the closed-loop system is internally stable.
1. If \( G(s) \) has a RHP zero at \( s=z \), then \( T(z)=0, S(z)=1 \)
2. If \( G(s) \) has a RHP pole at \( s=p \), then \( S(p)=0, T(p)=1 \)

**Proof.** For internal stability, \( S \) must be stable, hence it cannot have any RHP pole. Consequently, \( SF \), stable implies that \( F \) can’t have any RHP pole either, so \( F(z) \) must be finite. Thus, \( L(z)=G(z)F(z)=0 \), so

\[ T(z)=L(z)/(1+L(z))=0, \quad S(z)=1-T(z)=1. \]

Similarly, a RHP pole at \( s=p \) requires that \( S \) has a RHP zero at \( s=p \) (otherwise, \( SG \) would not be stable), so \( S(p)=0 \) and \( T(p)=1-S(p)=1. \)

The maximum modulus principle

**Theorem.** Suppose that \( f(s) \) is stable. Then the maximum value of \( |f(s)| \) for \( s \in \text{RHP} \) is attained along the imaginary axis, i.e.

\[ \|f\|_\infty = \sup_{\omega} |f(i\omega)| \geq |f(s_0)| \quad \forall s_0 \in \text{RHP} \]

**Proof.** See course on complex analysis.
Limitations from RHP zeros

**Theorem.** Let $W_s$ be stable and minimum phase, and let $S$ be the sensitivity of an internally stable closed-loop system. Then
\[
|W_s S|_\infty \leq 1 \Rightarrow |W_s(z)| \leq 1 \quad (|W_s(z)|^{-1} \geq 1)
\]
for every RHP zero $z$ of the loop gain $L=G_F$.

**Proof.** By the maximum modulus principle and interpolation constraints
\[
1 \geq |W_s S|_\infty \geq |W_s(z)S(z)| = |W_s(z)| \quad \forall z \in \text{RHP}
\]

Bandwidth limitation from RHP zero

Consider the weight $W_s(s) = \frac{1}{M_s} + \frac{\omega BS}{s}$ then
\[
|W_s(z)| \leq 1 \Rightarrow \frac{1}{M_s} + \frac{\omega BS}{z} \leq 1
\]
So
\[
\omega BS \leq \left(1 - M_s^{-1}\right) z < z
\]
The reasonable value $M_s=2$ gives the rule of thumb
\[
\omega BS \leq \frac{z}{2}
\]

Re-statement and interpretation

It is not possible to find a controller which
a) gives an internally stable closed-loop system, and
b) results in a sensitivity function $S$ that satisfies
\[
|S(\omega)| \leq |W_s^{-1}(\omega)| = M_s \left| \frac{i\omega}{\omega + \omega BS M_b} \right| \quad \forall \omega
\]
unless $\omega BS \leq (1 - M_s)^{-1}z$ for every RHP zero of $G(s)$

Example

Let $G(s) = \frac{1 - s}{s(s + 1)}$, $F_y(s) = \frac{s + 1}{a_0 s + a_1}$
If $a_0 = 1/(2\omega^2)$, $a_1 = (\omega + 1)/\omega$ then $S$ has poles in $\omega(-1 \pm i)$

$S$ for $\omega = 0.25, 0.5, 2, 8$ - pushing bandwidth results in peaking
Bandwidth limitations by time delays

Since

\[ e^{-sT} \approx \frac{1 - sT/2}{1 + sT/2} \]

a system with time delay \( T \)

\[ G(s) = G_0(s)e^{-sT} \]

can be seen as a system with a RHP zero at \( s = 2/T \).

Then, \( M_s = 2 \) suggests

\[ \omega_{BS} \leq \frac{1}{T} \]

Limitations from RHP poles

**Theorem.** Let \( W \) be stable and minimum phase, and let \( T \) be the complementary sensitivity of a stable closed-loop system. Then

\[ \|W_T\|_\infty \leq 1 \Rightarrow |W_T(p)| \leq 1 \]

for every RHP pole \( p \) of the loop gain \( L = F(s)G(s) \).

**Proof.** Similarly to the S-constraints, we have

\[ 1 \geq \|W_T\|_\infty \geq |W_T(p)T(p)| = |W_T(p)| \]

where the second inequality follows from maximum modulus and the final equality is due to the interpolation constraints.

Example

Let \( G(s) = \frac{s + 1}{s(s - 1)} \), \( F_y(s) = \frac{b_0s + b_1}{s + 1} \)

If \( b_0 = 1 + 2\omega, \quad b_1 = 2\omega^2 \), then \( T \) has poles in \( \omega(-1 \pm i) \)

\( T \) for \( \omega = 0.25, 0.5, 2, 8 \) - too low bandwidth forces \( T \) to peak
Implications for loop shaping

RHP zeros \( z \) and RHP poles restrict the bandwidth of the loop gain.

Would like bandwidth smaller than \( \frac{z}{2} \), larger than \( 2p \) (typically \( z > p \)).

Example: balancing act

Balancing a rod: \( G(s) = \frac{-g}{s^2(Mls^2 - (M + m)g)} \)

Where \( M, m \) are the masses of the hand and rod, respectively; \( l \) the length of the rod, and \( g \) is acceleration due to gravity.

Unstable pole at \( p = \sqrt{\frac{(M + m)g}{MI}} \)

With \( M=m \) and \( l=1 \) m, then \( p=4.5 \text{ rad/s} \)

Requires response time of 0.1-0.2 s

Balancing act cont’d

Try to balance the rod while only observing its base

\[ G(s) = \frac{ls^2 - g}{s^2(Mls^2 - (M + m)g)} \]

Introduces RHP zero at \( z = \sqrt{g/l} \)

Practically impossible to balance when \( M=m \), since \( z < p = \sqrt{2s} \)

Try!

Example: X-29

Under one flying condition, the X-29 can be modelled by

\[ G(s) = G(s)^2 - 26 = -6 \]

RHP pole at \( s=6 \rightarrow \omega_{LT} \geq 2 \times 6 = 12 \)

RHP zero at \( s=26 \rightarrow \omega_{BS} \leq 26/2 = 13 \)

Difficult to design a controller that satisfies these requirements!
Bode’s relations

Links phase and amplitude curves of loop gain

\[ \arg L(i\omega) \leq \frac{\pi}{2} \frac{d}{d\log \omega} \log |L(i\omega)| \]

A positive phase margin requires \( \arg L(i\omega_c) > -\pi \) so the negative slope of \( |L| \) can be at most 2 around cross-over \( \omega_c \).

Implications for loop transfer function

For small \( \omega \), it approximately holds that

\[ |S| < \epsilon \iff |L| > \frac{1}{\epsilon}, \quad |T| < \epsilon \iff |L| < \epsilon \]

→ Need sufficient spacing between frequency range where \( S \) is small and frequency range where \( T \) is small!

Bode’s integral theorem

**Theorem.** Suppose that \( L(s) = F_y(s)G(s) \) has relative degree \( \geq 2 \), and that \( L(s) \) has \( N_p \) RHP poles located at \( s = p_i \). Then, for closed-loop stability, the sensitivity function must satisfy

\[ \int_0^{\infty} \log |S(i\omega)| \, d\omega = \pi \sum_{i=1}^{N_p} \text{Re}(p_i) \]

Interpretation of Bode’s integral

All stable controllers give the same value of

\[ \int \log |S(i\omega)| \, d\omega \]

If \( L(s) \) is stable, then area for \( |S| \) above and below 1 is equal

- Sensitivity reduction in one frequency range comes at expense of sensitivity increase at another ("waterbed effect")

Unstable poles increase the overall sensitivity
Summary

Dynamics introduces fundamental limitations of feedback control performance
- RHP zero at \( z = \omega_{BS} \leq \frac{\pi}{2} \)
- Time delay \( T = \omega_{NS} \leq \frac{1}{T} \)
- RHP pole at \( p = \omega_{PT} \geq 2p \)

Bode’s relation
- good phase margin requires separation between frequency ranges where \( S \) is small and frequency ranges where \( T \) is small

Bode’s integral theorem
- reduced sensitivity in one frequency range comes at expense of higher sensitivity in other range ("waterbed effect")