2E1252
Control Theory and Practice

Lecture 4: Limitations and Conflicts

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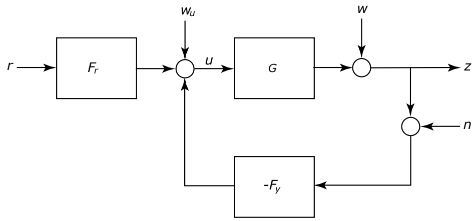
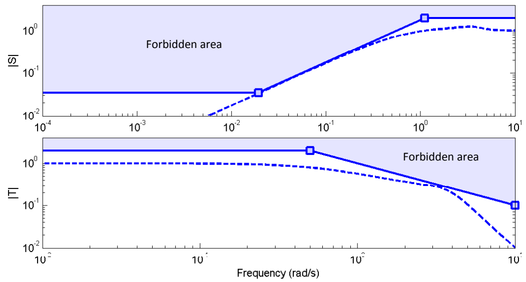
So far…



- Signal norms, system gains and the small gain theorem
- The closed-loop system and the design problem
  - Characterized by six transfer functions: need to look at all!
  - Internal stability: stability from all inputs to all outputs (sufficient to check that F_r, S, SG and SF_y are all stable)
  - Sensitivity function (suppression of load disturbances) and Complementary sensitivity (robust stability)

Frequency domain specifications



$$|S(i\omega)| \leq |W_S^{-1}(i\omega)|$$
$$|T(i\omega)| \leq |W_T^{-1}(i\omega)|$$

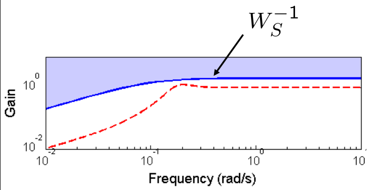
Can we choose weights W_S, W_T ("forbidden areas") freely?
- No, there are many constraints and limitations!

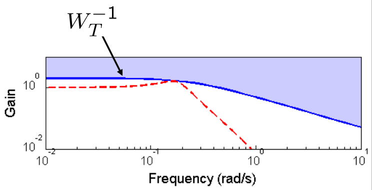
Today's lecture

- Fundamental limitations in control systems design
  - S+T=1 (both can't be small at the same time)
  - Can't attenuate disturbances at all frequencies
- Limiting factors:
  - Unstable poles
  - Non-minimum phase zeros
  - Time delays
  - Control authority (exercises; final part of course)
- Reasonable specifications, and rules-of-thumb!

Course book: Chapter 7.

Reasonable specifications





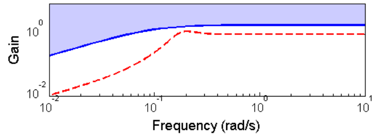
$\|SW_S\|_\infty \le 1$

implied by $|S(i\omega)| \le |W_S^{-1}(i\omega)|$

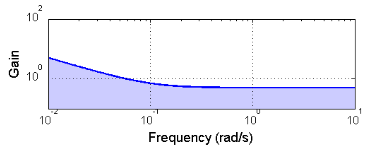
$\|TW_T\|_\infty \le 1$

implied by $|T(i\omega)| \le |W_T^{-1}(i\omega)|$

Specifications in terms of W_S



$$W_S^{-1} = M_S \frac{s}{s + \omega_{BS} M_s}$$



$$W_S = \frac{1}{M_S} + \frac{\omega_{BS}}{s} = \frac{1}{M_S}\frac{s + \omega_{BS} M_s}{s}$$

Interpolation constraints

Fact: Assume that the closed-loop system is internally stable.

1. If G(s) has a RHP zero at s=z, then T(z)=0, S(z)=1
2. If G(s) has a RHP pole at s=p, then S(p)=0, T(p)=1

Proof. For internal stability, S must be stable, hence it cannot have any RHP pole. Consequently, SF_y stable implies that F_y can't have any RHP pole either, so F(z) must be finite. Thus, $L(z)=G(z)F_y(z)=0$, so

T(z)=L(z)/(1+L(z))=0, S(z)=1-T(z)=1.

Similarly, a RHP pole at s=p requires that S has a RHP zero at s=p (otherwise, SG would not be stable), so S(p)=0 and T(p)=1-S(p)=1.

The maximum modulus principle

Theorem. Suppose that f(s) is stable. Then the maximum value of |f(s)| for s in the RHP is attained along the imaginary axis, i.e.

$$\|f\|_\infty = \sup_\omega |f(i\omega)| \ge |f(s_0)| \quad \forall s_0 \in \text{RHP}$$

Proof. See course on complex analysis.

Limitations from RHP zeros

Theorem. Let W_S be stable and minimum phase, and let S be the sensitivity of an internally stable closed-loop system. Then

$$\|W_S S\|_\infty \leq 1 \Rightarrow |W_S(z)| \leq 1 \qquad \left(|W_S(z)^{-1}| \geq 1\right)$$

for every RHP zero z of the loop gain $L=GF_y$.

Proof. By the maximum modulus principle and interpolation constraints

$$1 \geq \|W_S S\|_\infty \geq |W_s(z)S(z)| = |W_S(z)| \quad \forall z \in \text{RHP}$$

Bandwidth limitation from RHP zero

Consider the weight

$$W_S(s) = \frac{1}{M_s} + \frac{\omega_{BS}}{s}$$

then

$$|W_S(z)| \leq 1 \Rightarrow \frac{1}{M_s} + \frac{\omega_{BS}}{z} \leq 1$$

So

$$\omega_{BS} \leq \left(1 - M_s^{-1}\right) z < z$$

The reasonable value M_s=2 gives the rule of thumb

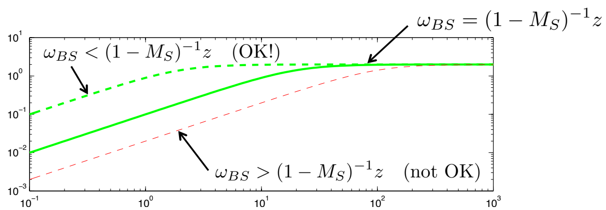
$$\boxed{\omega_{BS} \leq \frac{z}{2}}$$

Re-statement and interpretation

It is not possible to find a controller which
a) gives an internally stable closed-loop system, and
b) results in a sensitivity function S that satisfies

$$|S(i\omega)| \leq |W_S^{-1}(i\omega)| = M_S \left|\frac{i\omega}{i\omega + \omega_{BS} M_S}\right| \qquad \forall \omega$$

unless $\omega_{BS} \leq (1 - M_S)^{-1} z$ for every RHP zero of G(s)

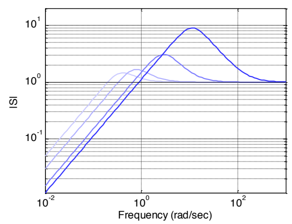


Example

Let $G(s) = \frac{1 - s}{s(s+1)}$, $\quad F_y(s) = \frac{s+1}{a_0 s + a_1}$

If $a_0 = 1/(2\omega^2)$, $\quad a_1 = (\omega + 1)/\omega$ then S has poles in $\omega(-1 \pm i)$

S for ω=0.25, 0.5, 2, 8 – pushing bandwidth results in peaking



Bandwidth limitations by time delays

Since

$$e^{-sT} \approx \frac{1 - sT/2}{1 + sT/2}$$

a system with time delay T

$$G(s) = G_0(s)e^{-sT}$$

can be seen as a system with a RHP zero at s=2/T.

Then, M_s=2 suggests

$$\boxed{\omega_{BS} \leq \frac{1}{T}}$$

Limitations from RHP poles

Theorem. Let W_T be stable and minimum phase, and let T be the complementary sensitivity of a stable closed-loop system. Then

$$\|W_T T\|_\infty \leq 1 \Rightarrow |W_T(p)| \leq 1$$

for every RHP pole p of the loop gain $L=F_yG$

Proof. Similarly to the S-constraints, we have

$$1 \geq \|W_T T\|_\infty \geq |W_T(p)T(p)| = |W_T(p)|$$

where the second inequality follows from maximum modulus and the final equality is due to the interpolation constraints.

Bandwidth limitation from RHP pole

Consider the weight

$$W_T(s) = \frac{s}{\omega_{0T}} + \frac{1}{M_T}$$

then

$$|W_T(p)| \leq 1 \Rightarrow \frac{p}{\omega_{0T}} + \frac{1}{M_T} \leq 1$$

So

$$\omega_{0T} \geq \frac{p}{1 - 1/M_T} \geq p$$

The more reasonable value M_T=2 gives the rule of thumb

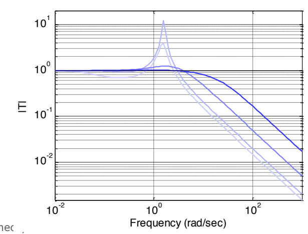
$$\boxed{\omega_{0T} \geq 2p}$$

Example

Let $G(s) = \dfrac{s+1}{s(s-1)}, \quad F_y(s) = \dfrac{b_0 s + b_1}{s+1}$

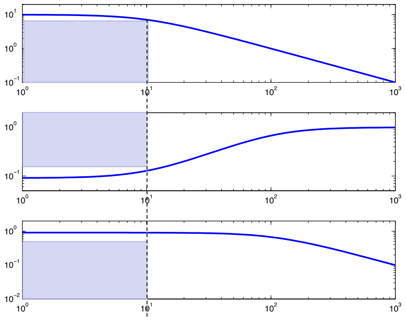
If $b_0 = 1 + 2\omega, \quad b_1 = 2\omega^2$ then T has poles in $\omega(-1 \pm i)$

T for ω=0.25, 0.5, 2, 8 – too low bandwidth forces T to peak



Implications for loop shaping

RHP zeros z and RHP poles restrict the bandwidth of the loop gain

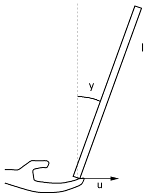


Would like bandwidth smaller than z/2, larger than 2p (typically z>>p)

Example: balancing act

Balancing a rod: $G(s) = \frac{-g}{s^2(Mls^2 - (M+m)g)}$

Where M, m are the masses of the hand and rod, respectively; l the length of the rod, and g is acceleration due to gravity.



Unstable pole at

$$p = \sqrt{\frac{(M+m)g}{Ml}}$$

With M=m, l=1 m, then p=4.5 rad/s

Requires response time of 0.1-0.2 s

Balancing act cont'd

Try to balance the rod while only observing its base

$$G(s) = \frac{ls^2 - g}{s^2(Mls^2 - (M+m)g)}$$

Introduces RHP zero at $z = \sqrt{g/l}$

Practically impossible to balance when M=m, since $z < p = \sqrt{2}z$

Try!

Example: X-29

Under one flying condition, the X-29 can be modelled by

$$G(s) = \hat{G}(s)\frac{s-26}{s-6}$$

RHP pole at s=6 → $\omega_{0T} \geq 2 \times 6 = 12$
RHP zero at s=26→ $\omega_{BS} \leq 26/2 = 13$

Difficult to design a controller that satisfies these requirements!

Bode's relations

Links phase and amplitude curves of loop gain

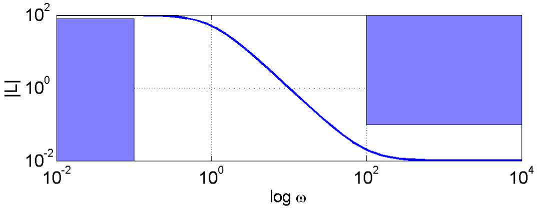
$$\arg L(i\omega) \leq \frac{\pi}{2}\frac{d}{d\log\omega}\log|L(i\omega)|$$

A positive phase margin requires $\arg L(i\omega_c) > -\pi$ so the negative slope of |L| can be at most 2 around cross-over ω_c

Implications for loop transfer function

For small ω, it approximately holds that

$$|S| < \epsilon \Leftrightarrow |L| > \frac{1}{\epsilon}, \qquad |T| < \epsilon \Leftrightarrow |L| < \epsilon$$



→Need sufficient spacing between frequency range where S is small and frequency range where T is small!

Bode's integral theorem

Theorem. Suppose that L(s)=F_y(s)G(s) has relative degree $\geq$ 2, and that L(s) has N_p RHP poles located at s=p_i. Then, for closed-Loop stability, the sensitivity function must satisfy

$$\int_0^\infty \log|S(i\omega)|\, d\omega = \pi \sum_{i=1}^{N_p} \mathrm{Re}(p_i)$$

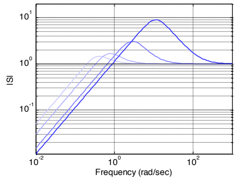
Intepretation of Bode's integral

All stable controllers give the same value of

$$\int \log|S(i\omega)|\, d\omega$$

If L(s) is stable, then area for |S| above and below 1 is equal

- Sensitivity reduction in one frequency range comes at expense of sensitivity increase at another ("**waterbed effect**")



Unstable poles increase the overall sensitivity

Summary

Dynamics introduces fundamental limitations of feedback control performance

- RHP zero at z $\Rightarrow \omega_{BS} \leq z/2$
- Time delay T $\Rightarrow \omega_{BS} \leq 1/T$
- RHP pole at p $\Rightarrow \omega_{0T} \geq 2p$

Bode's relation

- good phase margin requires separation between frequency ranges where S is small and frequency ranges where T is small

Bode's integral theorem

- reduced sensitivity in one frequency range comes at expense of higher sensitivity in other range ("waterbed effect")