Schedule: KTH-social Anmälan projekt, Inlämning rapport 2
kth social, next lecture... (Improve KTH-social

Repetition:

Signal norms – How big is a signal?

System norms – How much larger can the output be compared to the input?

$$\|u(t)\|_2 = \sqrt{\int_{-\infty}^{\infty} |u(t)|^2} \quad \text{– (energy norm)}$$

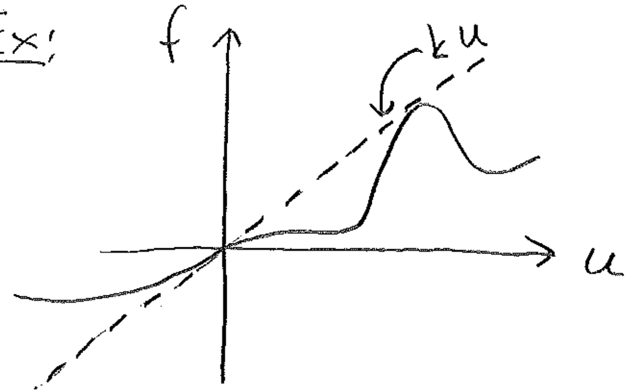
$$\|G\| = \sup_{\omega} |G(i\omega)| \quad \text{– (Linear systems)}$$

$$\|f\| = \sup_{u(t)} \frac{\|f(u(t))\|_2}{\|u(t)\|_2} \quad \text{– (static nonlinearity)}$$

Useful observation:

If $|f(u)| \leq |ku|$ then $\|f\| \leq |k|$

Ex:

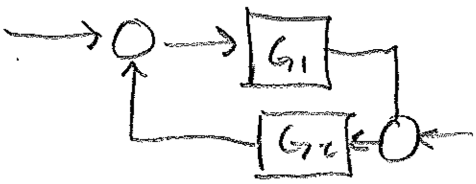


Small gain theorem:

If G_1 and G_2 are stable and $\|G_1\| \cdot \|G_2\| < 1$,

Then the closed loop is stable. For linear systems we require

$$\|G_1 G_2\| < 1$$



Internal stability:

All signals must remain bounded if we inject a bounded disturbance anywhere in the system.

(Enough to check S, SG, SF_y, F_r)

Theory: Limitations and Robustness

Limitations: Cannot always do what we want.

RHP-zeros $(z-s)$: Bandwidth limitation $\omega_{BS} < \frac{z}{2}$

Time delay $e^{-\tau s}$: Bandwidth limitation $\omega_c < \frac{1}{\tau}$

RHP-pole $(p-s)$: Minimum Bandwidth $\omega_B > 2p$

Bode's relation: Phase and amplitude are connected.

$$\arg(G(i\omega)) \lessapprox \frac{\pi}{2} \cdot \underbrace{\text{slope } \log|G(i\omega)|}_{\text{Slope of the amplitude curve in the Bode diagram.}}$$

(Exact relation can be found in the book.)

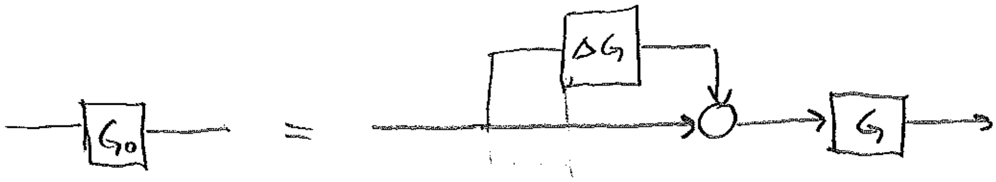
Robustness: Models are imperfect. We need to design controllers that work even if there are model errors.

Model error: G - The model

G_0 - The true system

We can often express the true model as

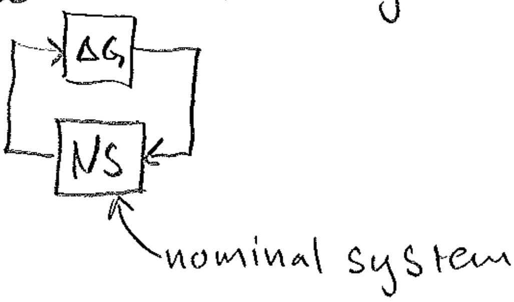
$$G_0 = (1+\Delta G)G \,, \quad \Delta G \text{ - relative model error.}$$



What is ΔG? We typically don't know!

$\Rightarrow$ Require the system to behave well for a large set of errors, e.g. all stable ΔG such that $\|\Delta G\| < 1$.

Stability: We can often redraw the system and use the small gain theorem.



7.1 RHP-zero:

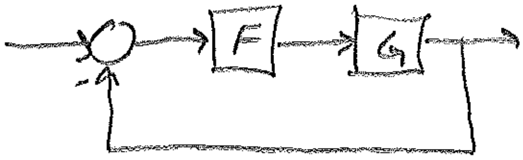
Given $G(s) = \dfrac{s-3}{s+1}$

We want $T(s) = \dfrac{5}{s+5}$

Using the thumbrule: $\omega_B \le \frac{3}{2}$, $\omega_B = 5$

a) Compute an F that gives this T. Will it work?

The system is



The loop gain is

$$L = FG \Rightarrow F = LG^{-1} \quad (1)$$

We know that

$$T = \frac{L}{1+L} \Rightarrow L = \frac{T}{1-T} \quad (2)$$

$$(1) \,\&\, (2) \Rightarrow F = \frac{T}{1-T} \cdot \frac{1}{G} = \frac{\frac{5}{s+5}}{1-\frac{5}{s+5}} \cdot \frac{s+1}{s-3} = \frac{5(s+1)}{s(s-3)}$$

Controller unstable with pole at $s=3$.

Internally stable? (Check S, SF_y, SG, F_r)

$$\text{Consider } SF_y = SF = F(1-T) = \frac{5(s+1)}{s(s-3)} \cdot \frac{s}{s+5} = \frac{5(s+1)}{(s-3)(s+5)}$$

RHP pole

$\Rightarrow$ Control input grows unbounded for small changes in the reference. Not internally stable.

b, The problem in a, originated from F beeing unstable, (due to cancelling the RHP-zero in G) Can we avoid that?

We know that $T = \frac{GF}{1+GF}$, hence any zero in G or F will be a zero in T unless it is cancelled. We therefore try to let the RHP-zero remain in T. This changes the bandwidth so we also add a pole to compensate.

$$T_2 = \frac{5}{(s+5)}\frac{(s-3)}{(s+3)}$$

The part $\frac{(s-3)}{(s+3)}$ is an all-pass filter.

Since $\left|\frac{i\omega-3}{i\omega+3}\right| = 1 \ \forall \omega$, the amplitude of all frequencies are untoched, only the phase is affected.

Hence $|T_2(i\omega)| = |T(i\omega)| \Rightarrow$ same bandwidth.

As in a, we get

$$F = \frac{T_2}{1-T_2}\cdot\frac{1}{G} = \frac{5(s-3)}{s^2+3s+30}\cdot\frac{(s+1)}{(s-3)} = \frac{5(s+1)}{s^2+3s+30}$$

Here F is stable! (it is enough that the coefficients are positive for a second order system)

Internal stability:

Check S, SF_y, SG, F_r

Controller is stable $\Rightarrow$ enough to check SG (see 6,2 in swedish course book).

$$SG = (1 - T_2)G = \frac{s^2+3s+30}{s^2+8s+15} \cdot \frac{s-3}{s+1} = \frac{(s^2+3s+30)(s-3)}{(s+5)(s+3)(s+1)}$$

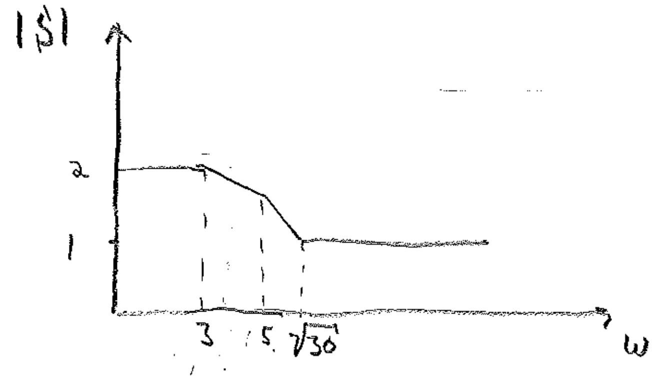
All poles stable $\Rightarrow$ Internally stable.

7.1 c) $S = (1-T) = \frac{s^2+3s+30}{s^2+8s+15}$

How does this look? Asymptotic scetching...

Breakpoints at ~~$\omega = \sqrt{15}$~~ & $\omega = \sqrt{30}$, $S(0) = 2$

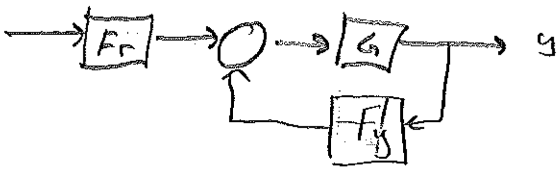
$\omega = 3$, $\omega = 5$



This basically means that we amplify low frequency disturbances and let high frequency disturbances pass through undamped.

This is an example of a controller which behaves opposite of what we would like when it comes to disturbances!

7.1 d) We use a two degrees of fredeom control structure.



Could we get stable F_r & F_y such that

$$G_c = \frac{GF_r}{1+GF_y} = \frac{5}{s+5}, \quad \text{with } G = \frac{s-3}{s+1} \ ?$$

does not have a pole at s=3

No! Assume F_r & F_y are stable, then

$F_r(3)$ & $F_y(3)$ are finite, but $G(3)=0$ $\Rightarrow$

$$G_c(3) = \frac{0 \cdot F_r(3)}{1 + 0 \cdot F_y(3)} = 0 \neq \frac{5}{3+5}$$

7.1 Remarks:

A rule of thumb is that we cannot normally make the bandwidth larger than about $\omega_B \approx \frac{z}{2}$ where z is the RHP zero.

In this exercise we tried to get it to be $\omega_B = 5$ but had a zero at $z = 3$!!

By considering limiting factors such as time delays, RHP-zeros etc. we can often make a judgement if a set of specifications are feasible at all before spending lots of time trying to do something impossible.

7.2 We have a system with a 1 second time delay and a RHP-zero at $z=3$.

The open loop amplitude is decreasing, i.e., $|L(i\omega)| \searrow$

What is the highest ω_c we can get?

We assume that we want a stable system using a stable, causal controller.

We can write the system as

$$G(s) = \underset{\text{RHP-zero}}{(3-s)}\ \underset{\text{time delay}}{e^{-s}}\ \underset{\text{Remaining dynamics assumed to be invertable.}}{\tilde{G}(s)}$$

We require a stable system $\Rightarrow$

$\varphi_m > 0$. $\varphi_m = \pi + \arg(L(i\omega_c))$

<u>We need $L(i\omega_c)$:</u>

$L(s) = G(s)F(s) = (3-s)e^{-s}\tilde{G}(s)F(s)$.

$\tilde{G}(s)$ - can invert in the controller.

$(3-s)$ - cannot invert (unstable controller).

e^{-s} - cannot invert (non causal controller).

$\Rightarrow$

$L(s) = (3-s)e^{-s}\hat{L}(s)$

However, plugging this $L(s)$ into the expression for the phase margin does not say much since we don't know much about $\hat{L}(s)$

We rewrite this as

$$L(s) = \underbrace{\frac{3-s}{3+s} e^{-s}}_{\text{non minimum phase part}} \underbrace{(3+s)\tilde{L}(s)}_{\text{minimum phase part}} = \frac{3-s}{3+s} e^{-s} L_{mp}(s)$$

Why? Since $\left| \frac{3-i\omega}{3+i\omega} e^{-i\omega} \right| = 1 \quad \forall \omega$ we get

all-pass   time delay

$$|L(i\omega)| = |L_{mp}(i\omega)| \Rightarrow |L_{mp}(i\omega)| \text{ is decreasing}$$

Hence we know it has slope ≤ 0 and by Bode's relation we get

$$\arg |L_{mp}(i\omega_c)| \lesssim \frac{\pi}{2} \cdot \underbrace{\text{slope } \log |L_{mp}(i\omega_c)|}_{\leq 0} \leq 0$$

Inserting this into the expression for φ_m we get

$$\varphi_m = \pi + \arg\left(\frac{3-i\omega_c}{3+i\omega_c} e^{-i\omega_c}\right) + \underbrace{\arg(L_{mp}(i\omega_c))}_{\leq 0} \Rightarrow$$

$$\varphi_m \leq \pi + \arg\left(\frac{3-i\omega_c}{3+i\omega_c}\right) + \arg(e^{-i\omega_c}) = \pi - 2\tan^{-1}\frac{\omega_c}{3} - \omega_c$$

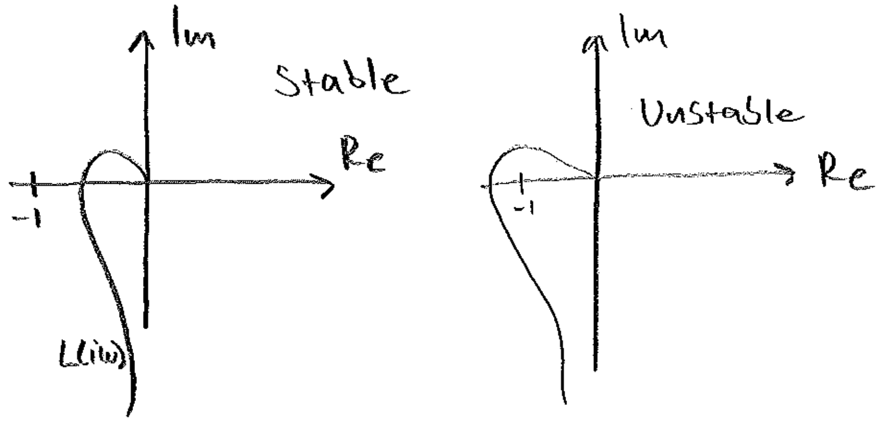
From $\varphi_m > 0$ we get a limit

$$0 = \pi - 2\tan^{-1}\frac{\omega_c}{3} - \omega_c \Rightarrow \omega_c \approx 2$$

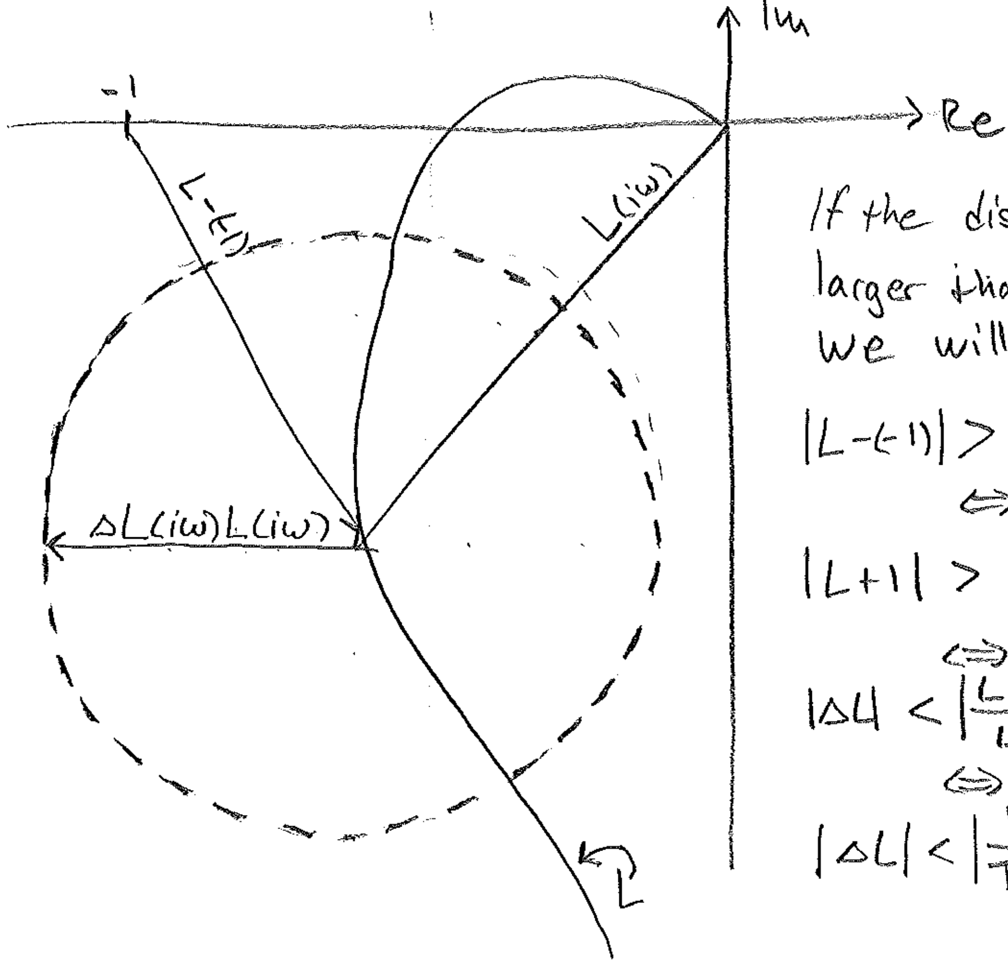
<u>Remark</u>: Note that this is an <u>optimistic</u> bound stating that $\omega_c > 2 \Rightarrow$ instability

However $\omega_c < 2 \not\Rightarrow$ stability, 6.5 According to the Nyquist criteria, a closed-loop system is stable if the curve $L(i\omega)$ is to the "right" of the point -1 as ω runs through the positive real axis.

Ex



Assume the nominal system is stable and consider what happens for $L_p = L + \Delta L \cdot L$



If the distance to -1 from L is larger than the perturbation we will still be stable!

$$|L-(-1)| > |\Delta L \cdot L|$$

$$\Leftrightarrow$$

$$|L+1| > |\Delta L||L|$$

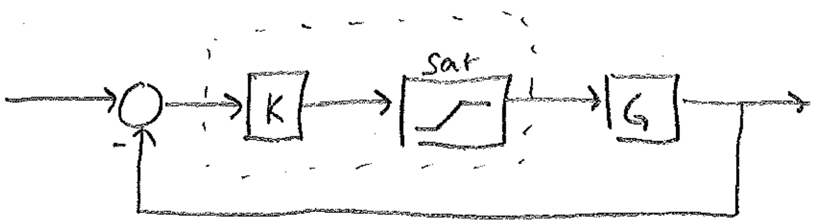
$$\Leftrightarrow$$

$$|\Delta L| < \left|\frac{L+1}{L}\right|$$

$$\Leftrightarrow$$

$$|\Delta L| < \left|\frac{1}{T}\right|$$

7.19 a) We have a system



where $\text{sat}(x) = \begin{cases} 1, & x \geq 1 \\ x, & |x| \leq 1 \\ -1, & x \leq -1 \end{cases}$ and $G(s) = \frac{1}{s+\varepsilon}$

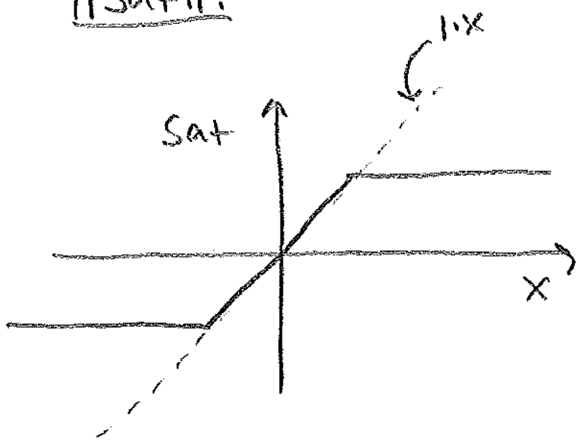
Derive a stability criterion on K using the small gain theorem.

If all systems are stable and $\|K\| \cdot \|\text{sat}\| \cdot \|G\| < 1$ then the closed loop is stable.

$\|K\| = |K|$ for a constant.

$\|G\| = \sup_{\omega} |G(i\omega)| = \sup_{\omega} \frac{1}{\sqrt{\omega^2+\varepsilon^2}} = \frac{1}{\varepsilon}$

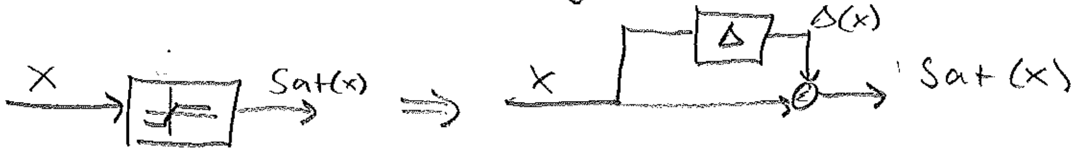
$\|\text{sat}\|$:



Since $|\text{sat}(x)| \leq |x| = |1 \cdot x|$ we know that $\|\text{sat}\| \leq 1$

Thus, stable if $|K| \cdot 1 \cdot \frac{1}{\varepsilon} < 1 \Rightarrow |K| < \varepsilon$

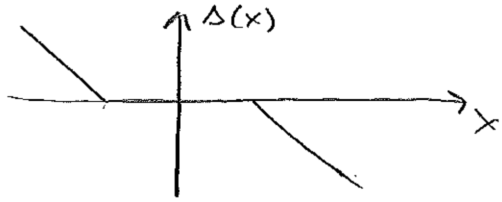
b) We want to describe the saturation as an uncertainty.



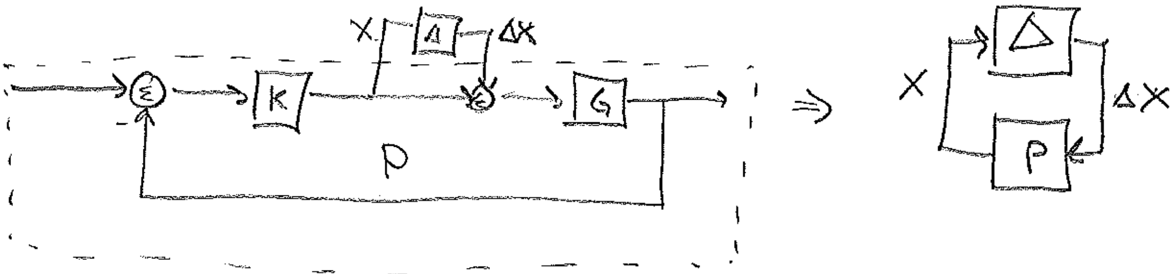
<u>What is Δ?</u>

$$\mathrm{Sat}(x) = x + \Delta(x) \Rightarrow \begin{cases} 1 = x + \Delta(x) & x \geq 1 \\ x = x + \Delta(x) & |x| \leq 1 \\ -1 = x + \Delta(x) & x \leq -1 \end{cases} \Rightarrow$$

$$\Delta(x) = \begin{cases} -x + 1 & x \geq 1 \\ 0 & |x| \leq 1 \\ -x - 1 & x \leq -1 \end{cases}$$



We redraw our system



<u>What is P?</u> We derive the transferfunction from $\Delta x \to x$

$$U(s) = -K G(s)(\Delta X(s) + X(s)) \Rightarrow$$

$$U(s) = \underbrace{\frac{-K G(s)}{1 + K G(s)}}_{P} \Delta X(s)$$

We use the small gain theorem

$\|\Delta\| \cdot \|P\| < 1$

$\|\Delta\|$: We have $|\Delta(x)| < |1 \cdot x|$ $\forall x \Rightarrow \|\Delta\| \leq 1$

The equality is attained since

$$\lim_{x \to \infty} \left|\frac{\Delta(x)}{x}\right| = \lim_{x \to \infty} \frac{x-1}{x} = 1$$

$\|P\|$: $$P(s) = \frac{\frac{-K}{s+\varepsilon}}{1+\frac{K}{s+\varepsilon}} = \frac{-K}{s+\varepsilon+K} \Rightarrow \text{stable if } K > -\varepsilon$$

$$\sup_{\omega} |P(i\omega)| = \left|\frac{-K}{\varepsilon+K}\right|$$

By the small gain theorem we have stability if $\left|\frac{K}{\varepsilon+K}\right| \cdot 1 < 1 \Rightarrow |K| < |\varepsilon + K|$

Which is true for $-\frac{\varepsilon}{2} < K < \infty$