Repetition:

Limitations:

RHP-zeros & time delays ⇒ Limited bandwidth

RHP-poles ⇒ Minimum bandwidth req.

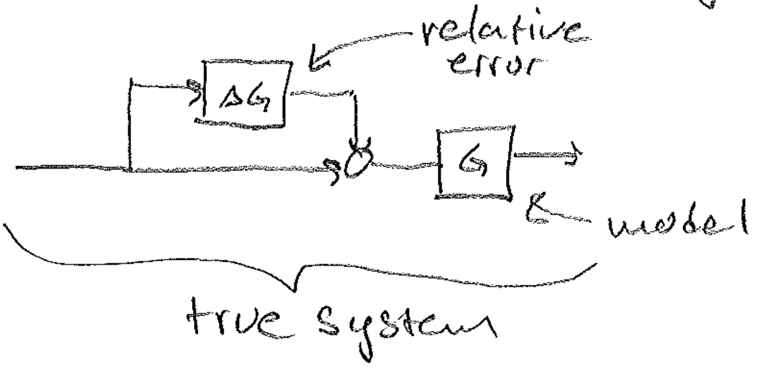
Bode's relation:

Phase & Amplitude are connected

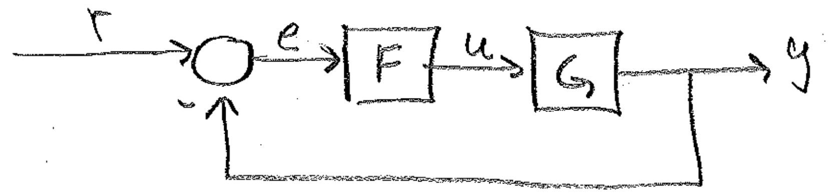
$$\arg(G(i\omega)) \lesssim \frac{\pi}{2} \cdot \underbrace{\text{slope } \log|G(i\omega)|}_{\frac{d \log|G(i\omega)|}{d \log \omega}} \quad \text{(approx...)}$$

Robustness:

Models are imperfect. Controller should work anyway!

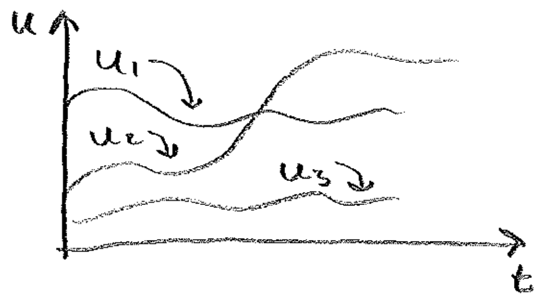


Theory: MIMO



For MIMO systems, signals are time-varying vectors.

ex: $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ We use column-vectors for signals.



Systems are described by transfer matrices:

If $u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ then $G = n \times m$ matrix

Ex: $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\Rightarrow$ $G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}$

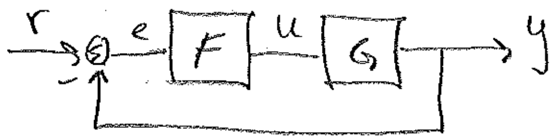
<u>Deriving transferfunctions</u>: Same way as for SISO <u>but</u>:

Matrices generally do not commute

$AB \neq BA$ (not even for square matrices)

$\Rightarrow FG \neq GF$ for MIMO systems.

<u>Ex</u>:



$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \qquad F = \begin{bmatrix} F_1 & F_2 \end{bmatrix} \Rightarrow u = u_1$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$$

The transfer function from $e \to y$ is

$$GF = \begin{bmatrix} G_1F_1 & G_1F_2 \\ G_2F_1 & G_2F_2 \end{bmatrix} \quad - \text{ 2x2 matrix}$$

and <u>not</u> $FG = F_1G_1 + F_2G_2$ - scalar

$\Rightarrow$ We need to be careful with order!

$r - y$

Statespace: Same as for SISO but B&C "Larger"

as many <u>columns</u> as inputs

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

as many <u>rows</u> as outputs

Going from Statespace to transfer matrix:

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\} \Rightarrow G(s) = C(sI-A)^{-1}B + D$$

<u>Other way is harder!</u>

Poles, state space:

For a minimal realization, the poles of a MIMO system is given by the eigenvalues of the A-matrix. (Same as SISO).

Observable & Controllable:

Same as in SISO-case.

$S = (B \ AB \ \cdots \ A^{n-1}B)$ - Controllability matrix

$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ - Observability matrix

If S & $\mathcal{O}$ has full rank, then the system is Controllable & Observable respectively.

Poles & zeroes from transfer matrices:

Minor = determinant of matrix when rows or columns are deleted.

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

$\det\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} = -3$ and $\det\begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} = -6$

are <u>second order</u> minors of A.

1, 2, 3, 4, 5, & 6 (the elements) are <u>first order</u> minors of A.

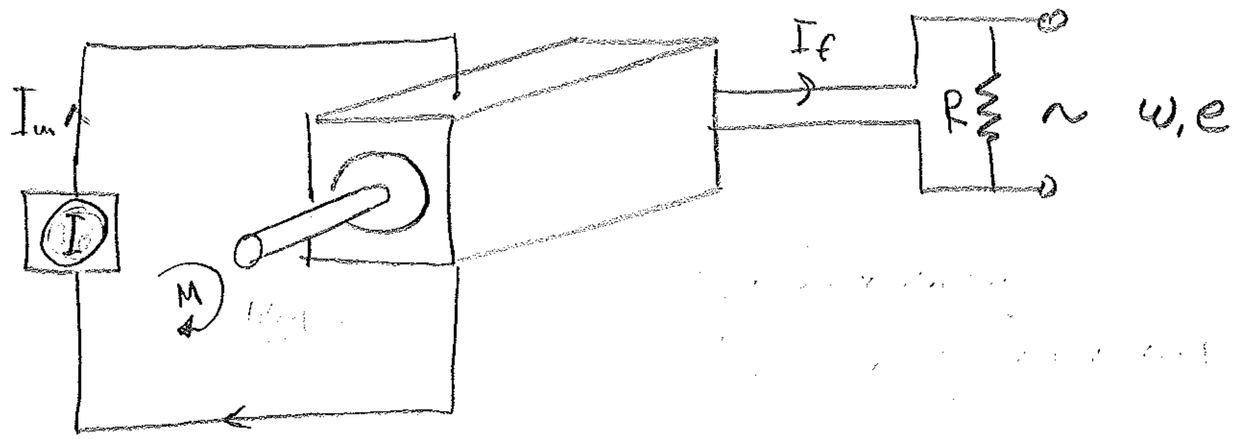
<u>Poles:</u> The polepolynomial are given as the Least Common Denominator (LCD) of <u>all</u> non-zero minors of $G(s)$.

<u>Zeroes:</u> The zero polynomial are given as the Greatest Common Divisor (GCD) of the non-zero maximum order minors of $G(s)$, normalized to have the pole polynomial as the denominator.

<u>Observation:</u> Need poles to compute zeroes.

2.3 Derive a transfer matrix.

We are given a simplified model of an AC-generator:



Signals: ω - frequency, e - peak output voltage } output

I_m - magnetising current, M - driving moment } Control input

R - load } disturbance

The signals are related by:

$$J\dot{\omega} = M - M_e$$
$$M_e = K_e \cdot \omega \cdot I_f$$
$$e = C_e \cdot I_m \cdot \omega$$
$$e = R \cdot I_f$$

I find a statespace representation

II Linearize the model about $\omega_0 = R_0 = I_{m0} = M_0 = 1$

III Derive the tranfer function matrix from $u = \begin{bmatrix} \Delta M \\ \Delta I_m \\ \Delta R \end{bmatrix}$ to $y = \begin{bmatrix} \omega \\ e \end{bmatrix}$

(I) We want express the system as:

$$\dot{x} = f(x,u)$$
$$y = h(x,u)$$

Where x is the state and u is the input.

<u>We are given:</u> $u = \begin{bmatrix} M \\ I_n \\ R \end{bmatrix}$, $y = \begin{bmatrix} \omega \\ e \end{bmatrix}$

<u>What is x?</u>

We only have one first order ODE $\Rightarrow$ we need one state. The ODE involves a derivative of ω so we let $x = \omega$.

<u>Derive f:</u>

$$e = I_f \cdot R \Rightarrow I_f = \frac{e}{R} \Rightarrow I_f = \frac{C_e \cdot I_m \cdot \omega}{R}$$

$$J\dot{\omega} = M - M_e = M - K_e \omega I_f = M - \frac{K_e C_e \cdot \omega^2 I_m}{R}$$

Inserting values for constants

$$J = K_e = C_e = 1 \Rightarrow$$

$$\dot{\omega} = M - \frac{\omega^2 I_m}{R} = f(\omega, u)$$

<u>Derive h:</u>

$$y = \begin{bmatrix} \omega \\ e \end{bmatrix} = \begin{bmatrix} \omega \\ C_e \cdot I_m \cdot \omega \end{bmatrix} = \begin{bmatrix} \omega \\ I_m \omega \end{bmatrix} = h(\omega, u)$$

(II) Linearize

Is $\omega_0 = R_0 = Im_0 = M_0 = 1$ a stationary solution?

$$f(\omega_0, u_0) = M_0 - \frac{\omega_0^2 Im_0}{R_0} = 1 - 1 = 0, \quad \text{OK!}$$

Introduce deviation variables

$$\Delta\omega = \omega - \omega_0 \qquad \Delta u = u - u_0 \qquad \Delta y = y - y_0$$

A Taylor expansion dropping higher order terms $\Rightarrow$

$$\Delta\dot{\omega} \approx \frac{\partial f(\omega_0, u_0)}{\partial \omega}\Delta\omega + \frac{\partial f(\omega_0, u_0)}{\partial u}\Delta u$$

$$\Delta y \approx \frac{\partial h(\omega_0, u_0)}{\partial \omega}\Delta\omega + \frac{\partial h(\omega_0, u_0)}{\partial u}\Delta u$$

The Jacobians are

$$\frac{\partial f}{\partial \omega} = -\frac{2\omega Im}{R} \qquad \frac{\partial f}{\partial u} = \begin{bmatrix} 1 & -\frac{\omega^2}{R} & \frac{\omega^2 Im}{R^2} \end{bmatrix}$$

$$\frac{\partial h}{\partial \omega} = \begin{bmatrix} 1 \\ Im \end{bmatrix} \qquad \frac{\partial h}{\partial u} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \omega & 0 \end{bmatrix}$$

The linearized model becomes

$$\Delta\dot{\omega} = \overbrace{-2}^{A}\Delta\omega + \overbrace{\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}}^{B}\Delta u$$

$$\Delta y = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{C}\Delta\omega + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{D}\Delta u$$

(III) Derive transfer matrix:

$$G(s) = C(sI-A)^{-1}B + D$$

$$G(s) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} (s+2)^{-1} \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} =$$

$$= \frac{1}{s+2} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} =$$

$$= \frac{1}{s+2} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} + \frac{1}{s+2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & s+2 & 0 \end{bmatrix} = \underline{\underline{\frac{1}{s+2} \begin{bmatrix} 1 & -1 & 1 \\ 1 & s+1 & 1 \end{bmatrix}}}$$

3.12 Determine the poles and zeros to the system

$$G(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{3}{s+2} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

For the poles we need all minors.

1:st order: (elements) $\frac{2}{s+1}, \frac{3}{s+2}, \frac{3}{s+2}, \frac{1}{s+1}, \frac{1}{s+1}, \frac{1}{s+1}$

2:nd order: (eliminate last column)

$$\begin{vmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{vmatrix} = \frac{2}{(s+1)^2} - \frac{3}{(s+1)(s+2)} = \frac{1-s}{(s+1)^2(s+2)}$$

(eliminate middle column)

$$\begin{vmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{vmatrix} = \frac{1-s}{(s+1)^2(s+2)}$$

(eliminate first column)

$$\begin{vmatrix} \frac{3}{s+2} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{vmatrix} = \frac{3}{(s+2)(s+1)} - \frac{3}{(s+2)(s+1)} = 0$$

Pole polynomial = Least Common Denominator (LCD) for all non-zero minors.

$LCD = (s+1)^2(s+2) \Rightarrow$ poles are $\{-1, -1, -2\}$

For the zeroes we need the non-zero minors of maximum order.

$$\frac{(1-s)}{(s+1)^2(s+2)}, \quad \frac{(1-s)}{(s+1)^2(s+2)}$$

zero-polynomial = Greatest Common Divisor (GCD)

$GCD = (1-s) \Rightarrow$ RHP-zero at $s=1$.

2.5 We are asked to find a state-space representation of

$$G(s) = \left[\frac{1}{(s+1)(s+2)} \quad \frac{s+3}{(s+1)(s^2+s+1)} \right]$$

This is typically harder than to go the other way! That is to find $G(s)$ from the state-space.

For SISO-systems, we can use canonical forms to do this.

For a single-output system we can also do this!

Extract the Least Common Denominator

$$G(s) = \frac{1}{(s+1)(s+2)(s^2+s+1)} \left[s^2+s+1 \quad (s+3)(s+2) \right] \Rightarrow$$

$$G(s) = \frac{1}{s^4+4s^3+6s^2+5s+2} \left[s^2+s+1 \quad s^2+5s+6 \right]$$

Consider the effect of each input separetely

$$Y(s) = \frac{s^2+s+1}{s^4+4s^3+6s^2+5s+2} U_1(s) + \frac{s^2+5s+6}{s^4+4s^3+6s^2+5s+2} U_2(s)$$

Using the observable canonical form for the SISO-systems $u_1 \to y$ and $u_2 \to y$ yield the same A & C matrices but different B matrices. Thus we can write

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2 = Ax + \underbrace{[B_1 \; B_2]}_{B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = Cx.$$

Where $\quad A = \begin{bmatrix} -4 & 1 & 0 & 0 \\ -6 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \\ -20 & 0 & 0 & 0 \end{bmatrix} \quad C = [1 \ 0 \ 0 \ 0]$

$$B_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 0 \\ 1 \\ 5 \\ 6 \end{bmatrix} \Rightarrow B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 5 \\ 1 & 6 \end{bmatrix}$$

3,2 a) Is the system stable?

$$\dot{x} = \underbrace{\begin{pmatrix} -1 & 1,5 \\ -3 & 3,5 \end{pmatrix}}_{A} x + \begin{pmatrix} -1 & 0,5 \\ -2 & 0,5 \end{pmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The poles of the system is given by the eigenvalues of the A matrix.

$\det(\lambda I - A) = 0 \Rightarrow$

$$\det \begin{bmatrix} s+1 & -1,5 \\ 3 & s-3,5 \end{bmatrix} = s^2 - 2,5s - 3,5 + 4,5 = s^2 - 2,5s + 1,5$$
$$= (s-1,5)(s-1)$$

The system is unstable.

b) Is the system feedback stabilizable?

If the system is controllable we can place the poles wherever we want, hence it is stabilizable.

Controllability matrix

$$S = (B \ AB) = \begin{pmatrix} -1 & 0,5 & -2 & 0,25 \\ -2 & 0,5 & -4 & 0,25 \end{pmatrix}$$

S has full rank $\Rightarrow$ System controllable $\Rightarrow$ stabilizable.