Lecture 5: Multivariable systems

So far...

- SISO control revisited:
  - Signal norms, system gains and the small gain theorem
  - The closed-loop system and the design problem
    - Characterized by six transfer functions: need to look at all!
    - Fundamental limitations and waterbed effect.
  - Loop shaping to satisfy sensitivity function specifications.

From now and on: MIMO

- Basic properties of multivariable systems
- State-space theory, state feedback and observers
- Decentralized and decoupled control
- Robust loop shaping
- \( H_2 \) and \( H_1 \) optimal control

Today’s lecture

- Basic properties of multivariable systems
- Transfer matrices
- Block diagram calculations
- Gains and directions
- The multivariable frequency response
- Poles and zeros

The final part of the course considers systems with constraints
Transfer matrices

The Laplace transform \( X(s) \) of a signal \( x(t) \) is defined by

\[
X(s) = \int_0^\infty x(t)e^{-st} \, dt
\]

Given a linear system

\[
\dot{x}(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t) + Du(t)
\]

and assuming \( u(t) = 0 \) for \( t < 0 \) and \( x(0) = 0 \),

\[
Y(s) = (C(sI - A)^{-1}B + D)U(s) = G(s)U(s)
\]

If the system has multiple inputs and outputs, \( Y \) and \( U \) are vector-valued and \( G(s) \) is a matrix (i.e., a matrix-valued function of \( s \)).

Quiz: transfer matrices

1. What is the transfer matrix \( G(s) \) for the system

\[
\begin{align*}
\dot{x}_1 &= -x_1 + u_1 \\
\dot{x}_2 &= -x_2 + u_2 \\
y_1 &= x_1 \\
y_2 &= x_2
\end{align*}
\]

How does \( G(s) \) change when

2. Input two also affects the first state: \( \dot{x}_1 = -x_1 + u_1 + u_2 \)

3. The second state also affects output one: \( y_1 = x_1 + x_2 \)

4. The second state influences the first: \( \dot{x}_1 = -x_1 - x_2 + u_1 \)

Answer and observations

1. \( G(s) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{s+1} \end{pmatrix} \)

Independent subsystems \( \rightarrow \) (block)diagonal transfer matrix

2., 3. \( G(s) = \begin{pmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+1} \end{pmatrix} \)

Couplings \( \rightarrow \) nondiagonal \( G(s) \). Different \( A, B, C \) can give same \( G(s) \)

4. \( G(s) = \begin{pmatrix} \frac{1}{s+1} & -\frac{1}{(s+1)^2} \\ 0 & \frac{1}{s+1} \end{pmatrix} \)

1.-4. have the same poles (eigenvalues of \( A \)). Hard to see from \( G(s) \)

Example: series connection

Linear system viewed as interconnected multivariable systems

\[
\begin{align*}
\begin{pmatrix} Z_1(s) \\ Z_2(s) \end{pmatrix} &= \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix} U(s), \\
Y(s) &= \begin{pmatrix} G_{21}(s) & G_{22}(s) \end{pmatrix} U(s).
\end{align*}
\]

We see that \( Y(s) = G_2Z(s) = G_2G_0(s)U(s) \).

Note that \( G_2G_0 \leq G_2G_0 - \text{order matters!} \)
Block diagram calculations

Have to be careful when manipulating block diagrams.

**Example.** Let \( w, u, n = 0 \) and derive transfer matrix from \( r \) to \( z \)

\[
z = Gu = G(F_r - F_y) \Rightarrow z + GF_yz = GF_r
\]

\[
z = (I + GF_y)^{-1}GF_r \quad (\neq (I + F_yG)^{-1}GF_r)
\]

Quiz: the closed-loop MIMO system

Determine the sensitivity and complementary sensitivity for the linear multivariable system

**Recall:** \( S \) is transfer matrix from \( w \rightarrow z \), \( T \) is transfer matrix from \( -n \rightarrow z \)

Additional question: What is the relation between \( S \) and \( T \)?

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Frequency response and system gain

For a scalar linear system \( G(s) \) driven by \( u(t) = \sin(\omega t) \),

\[
y(t) = |G(i\omega)| \sin(\omega t + \arg G(i\omega))
\]

(after transients have died out). So

\[
\frac{|Y(i\omega)|}{|U(i\omega)|} = |G(i\omega)|
\]

The system gain (cf. Lecture 1) is defined as

\[
\sup_{\omega} \frac{||y||_2}{||u||_2} = \sup \frac{||G(i\omega)||_2}{||G||_\infty}
\]

Attained for sinusoidal input with frequency \( \omega \) such that \( |G(i\omega)| = ||G||_\infty \)

**Q:** What are the corresponding results for multivariable systems?
Operator norm of linear mapping

Consider the linear mapping \( y = Ax \) (\( x, y, A \) complex-valued).

Since
\[
y^2 = |Ax|^2 = (Ax)^*Ax = x^*A^*Ax
\]
we have
\[
|x|^2 \lambda_{\text{min}}(A^*A) \leq |y|^2 \leq |x|^2 \lambda_{\text{max}}(A^*A)
\]
So
\[
\sigma(A) \leq \frac{|y|}{|x|} \leq \sigma(A)
\]
Where \( \sigma(A) = \sqrt{\lambda(A^*A)} \) are called the \textit{singular values} of \( A \).

The singular value decomposition

A matrix \( A \in \mathbb{C}^{m \times r} \) (with \( r < m, \text{rank}(A) = r \)), can be represented by its \textit{singular value decomposition (SVD)}:
\[
A = U \Sigma V^* = [u_1 \cdots u_r] \text{diag}(\sigma_i) [v_1 \cdots v_r]^* = \sum_{i=1}^r \sigma_i u_i v_i^*
\]

where
- The positive scalars \( \sigma_i \) are the \textit{singular values} of \( A \)
- \( v_i \) are the \textit{input singular vectors} of \( A \), \( V^*V = I \)
- \( u_i \) are the \textit{output singular vectors} of \( A \), \( U^*U = I \)

SVD interpretation

Interpretation: linear mapping \( y = Ax \) can be decomposed as
- compute coefficients of \( x \) along input directions \( v_i \)
- scale coefficients by \( \sigma_i \)
- reconstitute along output directions \( u_i \)

Since \( \sigma_1 \leq \cdots \leq \sigma_r \), an input in the \( v \) direction is amplified the most. It generates an output in the direction of \( u \) (typically different from \( v \)).

Example

\[
G(0) = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.85 & 0.53 & 1.62 & 0 & 0.53 & 0.85 \\ -0.53 & 0.85 & 0 & 0.62 & -0.85 & 0.53 \end{bmatrix}^*
\]

\[
x = \begin{bmatrix} 0.53 \\ -0.85 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1.62 \\ -0.85 \end{bmatrix}
\]

The multivariable frequency response

For a linear multivariable system $Y(s) = G(s)U(s)$, we have

$$Y(i\omega) = G(i\omega)U(i\omega)$$

Since this is a linear mapping,

$$\sigma(G(i\omega)) \leq \frac{|Y(i\omega)|}{|U(i\omega)|} \leq \sigma(G(i\omega))$$

with equality if $U(i\omega)$ parallel with corresponding input singular vector.

For example,

$$\frac{|Y(i\omega)|}{|U(i\omega)|} = \sigma(G(i\omega))$$

only if $U(i\omega)$ parallel with input singular vector corresponding to $\sigma$.

As for scalar systems, we can use Parseval’s theorem to find

$$\|y\|_2 \leq \|G\|_\infty \|u\|_2$$

where

$$\|G\|_\infty = \sup_{\omega} |G(i\omega)| = \sup_{\omega} \sigma(G(i\omega))$$

Note: Worst-case input is sinusoidal at the frequency that attains the supremum, but its components are appropriately scaled and phase shifted (as specified by the input singular vector of $\sigma$).

Note: the infinity norm computes the maximum amplifications across frequency (sup) and input directions ($\sigma$).

Example: heat exchanger

Objective: control outlet temperatures $T_c$, $T_H$ by manipulating the flows $q_C$, $q_H$.

Model:

$$\begin{pmatrix} \dot{T}_c \\ \dot{T}_H \end{pmatrix} = \frac{1}{(100s + 1)(2.5s + 1)} \begin{pmatrix} -19(5s + 1) & 18 \\ -18 & 19(5s + 1) \end{pmatrix} \begin{pmatrix} q_C \\ q_H \end{pmatrix}$$

Singular values of heat exchanger

System gain: $\|G\|_\infty = 31 \text{ dB}$ ($=37$)
**Heat exchanger steady-state**

\[ G(0) = \begin{bmatrix} -19 & 18 \\ -18 & 19 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 37 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \]

Singular value decomposition reveals

- Maximum effect input (input singular vector corresponding to \( \sigma \))
  \[ \frac{q_C}{q_H} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} T_C \\ T_H \end{bmatrix} = \begin{bmatrix} 37 \\ 37 \end{bmatrix} \]

- Minimum effect input (corresponding to \( \sigma \))
  \[ \frac{q_C}{q_H} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} T_C \\ T_H \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \]

**Heat exchanger step responses**

Input direction has dramatic effect! (agrees with physical intuition)

*Note:* large difference in time-scales!

**Singular values and bandwidths**

What is the bandwidth of the system?

No single value, but a range. Depends on input directions.

**Today’s lecture**

Basic properties of multivariable systems

- Transfer matrices
- Block diagram calculations
- Gains and directions
- The multivariable frequency response
- Poles and zeros
Poles

**Definition.** The *poles* of a linear system are the eigenvalues of the system matrix in a minimal state-space realization.

**Definition.** The *pole polynomial* is the characteristic polynomial of the A matrix, \( \lambda(s) = \det(sI - A) \).

Alternatively, the poles of a linear system are the zeros of the pole polynomial, i.e., the values \( p_i \) such that \( \lambda(p_i) = 0 \).

Poles cont’d

Since the transfer matrix is given by

\[
G(s) = C(sI - A)^{-1}B + D = \frac{1}{\det(sI - A)} r(s)
\]

where \( r(s) \) is a polynomial in \( s \) (see book for precise expression), the pole polynomial must be “at least” the least common denominator of the elements of the transfer matrix.

**Example:** The system

\[
G(s) = \begin{bmatrix}
\frac{2}{s+1} & \frac{3}{s+2} \\
\frac{1}{s+1} & \frac{2}{s+1}
\end{bmatrix}
\]

must (at least) have poles in \( s=-1 \) and \( s=-2 \).

Poles of multivariable systems

**Theorem.** The pole polynomial of a system with transfer matrix \( G(s) \) is the common denominator of all minors of \( G(s) \)

**Recall:** A minor of a matrix \( M \) is the determinant of a (smaller) square matrix obtained by deleting some rows and columns of \( M \)

**Example:** The minors of

\[
G(s) = \begin{bmatrix}
\frac{2}{s+1} & \frac{3}{s+2} \\
\frac{1}{s+1} & \frac{2}{s+1}
\end{bmatrix}
\]

are \( \frac{2}{s+1} \), \( \frac{3}{s+2} \), and \( \frac{1}{s+1} \) and \( \det G(s) = \frac{1-s}{(s+1)^2(s+2)} \)

Thus, the system has poles in \( s=-1 \) (a double pole) and \( s=-2 \).

Zeros

**Theorem.** The zero polynomial of \( G(s) \) is the greatest common divisor of the maximal minors of \( G(s) \), normed so that they have the pole polynomial of \( G(s) \) as denominator. The zeros of \( G(s) \) are the roots of its zero polynomial.

**Example:** The maximal minor of

\[
G(s) = \begin{bmatrix}
\frac{2}{s+1} & \frac{3}{s+2} \\
\frac{1}{s+1} & \frac{2}{s+1}
\end{bmatrix}
\]

is \( \det G(s) = \frac{1-s}{(s+1)^2(s+2)} \) (already normed!).

Thus, \( G(s) \) has a zero at \( s=1 \).
Quiz: multivariable poles and zeros

What are the poles and zeros of the multivariable system

\[ G(s) = \frac{1}{(s+1)(s-1)} \begin{pmatrix} 1 & 1 \\ s-1 & s+1 \end{pmatrix} \]

Notes on poles and zeros

For scalar system \( G(s) \) with poles \( p \) and zeros \( z \),

\[ G(z) = 0, \quad G(p) = \infty \]

For a multivariable system, directions matter!

For a system with pole \( p \), there exist vectors \( u_p, v_p \):

\[ u_p^*G(p) = \infty \quad G(p)v_p = \infty \]

Similarly, a zero at \( z \) implies the existence of vectors \( u_z, v_z \):

\[ u_z^*G(z) = 0 \quad G(z)v_z = 0 \]

As for scalar systems, a zero at \( s=z \) implies that there exists a signal on the form \( u(t) = v_z e^{-zt} \) for \( t \geq 0 \), and \( u(t) = 0 \) for \( t < 0 \), and initial values \( x(0) = x_z \) so that \( y(t) = 0 \) for \( t \geq 0 \).

Summary

An introduction to multivariable linear systems:

- Block diagram manipulations (order matters!)
- System gain (directions matter!)
- Poles and zeros

More next week!