EL2520
Control Theory and Practice

Lecture 7:
Multivariable loop shaping

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Modern loop shaping

The modern view of control systems analysis and design

- signal norms, system gains, frequency responses
- constraints on many transfer functions (cf. ”gang of six”)
- fundamental limitations, robustness

Extends from SISO to MIMO

Today’s lecture: modern loop shaping using H-infinity design

- Specify controller performance in terms of frequency responses
- Convert these constraints to weights
- Constrain norms of weighted transfer functions
- Directly design optimal controllers

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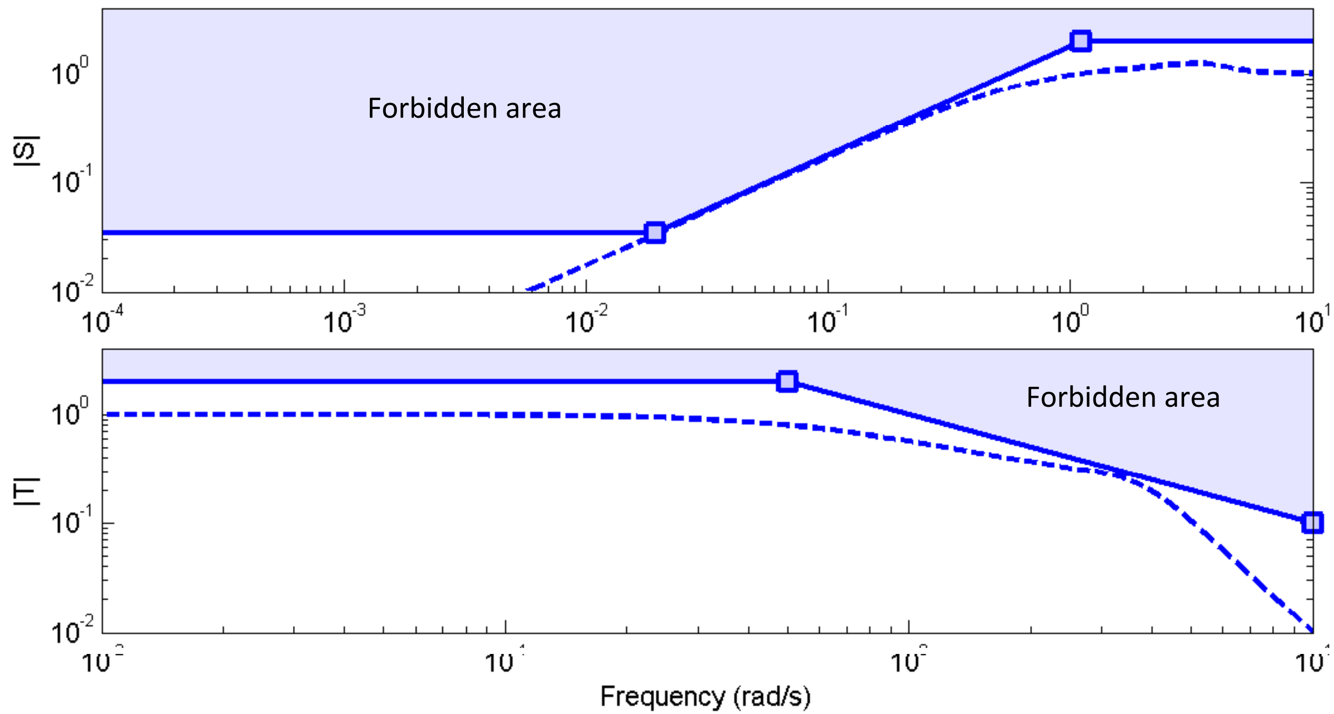
Sensitivity shaping

Convenient to design controller by constraining critical transfer matrices



$$\|W_S S\|_\infty \leq 1$$

$$\|W_T T\|_\infty \leq 1$$

$$\vdots$$

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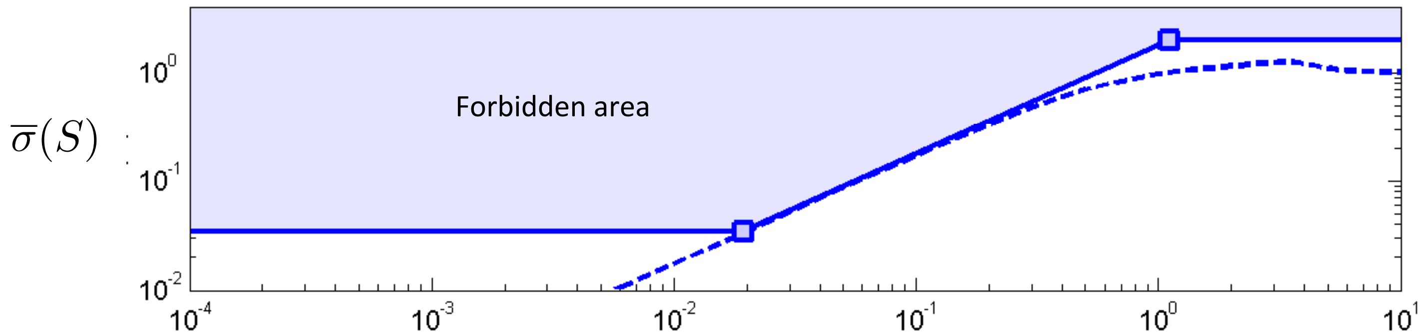
Constraining singular values

If we assume that W_S is scalar, then $\|W_S S\|_\infty \leq 1$ is implied by

$$|W_S(i\omega)| \cdot \overline{\sigma}(S(i\omega)) \leq 1 \qquad \forall\omega$$

Hence, W_S defines a “forbidden area” for the singular values.



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Fundamental limitations

As for SISO, non-minimum phase elements limit how we can select weights

Theorem. Let G have a zero z in the right half plane, and let the scalar transfer function W_S be stable and minimum phase. Then, a necessary condition for

$$\|W_S S\|_\infty = \sup_\omega \overline{\sigma}(W_S(i\omega)S(i\omega)) \leq 1$$

is that

$$|W_S(z)| \leq 1$$

Proof is analogous to SISO case (see book).

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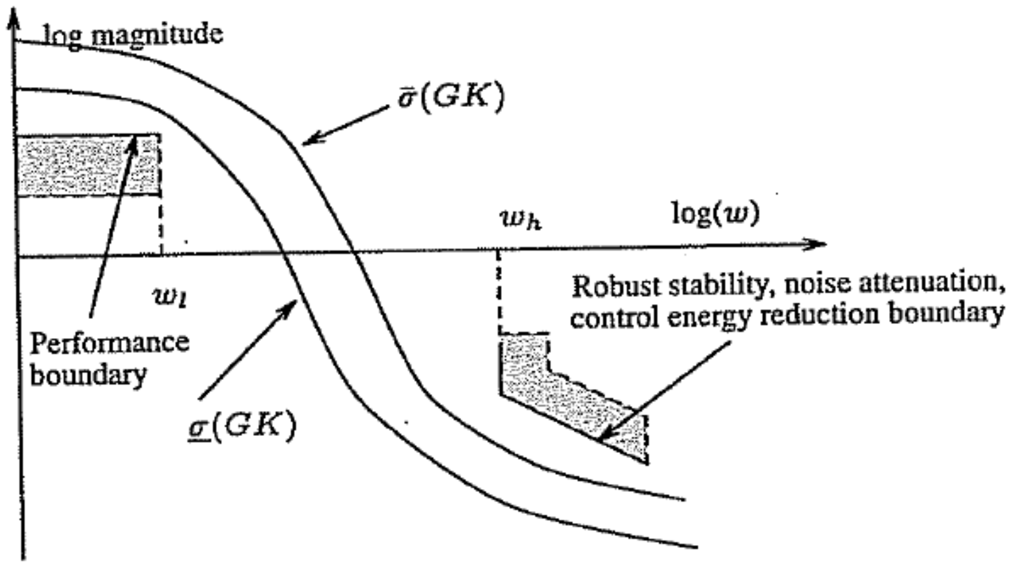
Multivariable loop shaping

As in SISO-case, constraints on S and T can be translated to constraints on the (**singular values** of the) loop gain L.



Hard to use for manual design of controllers (why?)

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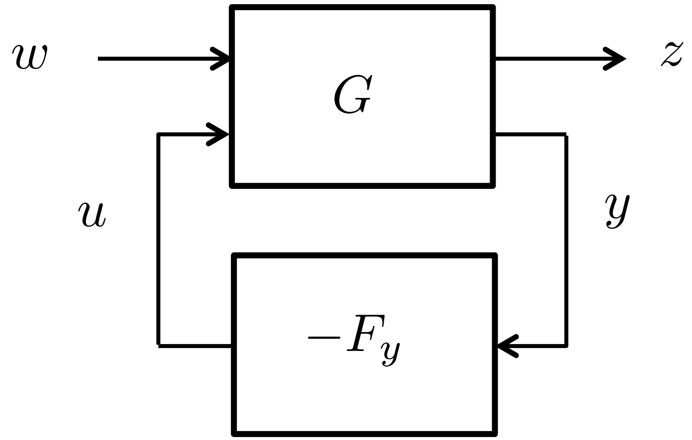
Optimization-based design

Sensitivity-shaping can be done using the following result:

Fact: It is possible to find a controller that ensures that the gain from w to z is less than γ whenever such a controller exists.



The controller is linear and of the same order as G.

Approximate design specifications

Problem: find controller that satisfies specifications

$$\begin{aligned} \|W_S S\| &\leq 1 \\ \|W_T T\| &\leq 1 \\ \|W_{SF_y} S F_y\| &\leq 1 \end{aligned}$$

If we consider the approximate constraint

$$\left\| \begin{pmatrix} W_S S \\ W_T T \\ W_{SF_y} S F_y \end{pmatrix} \right\| \leq 1$$

this can be viewed as the norm of an "extended system"

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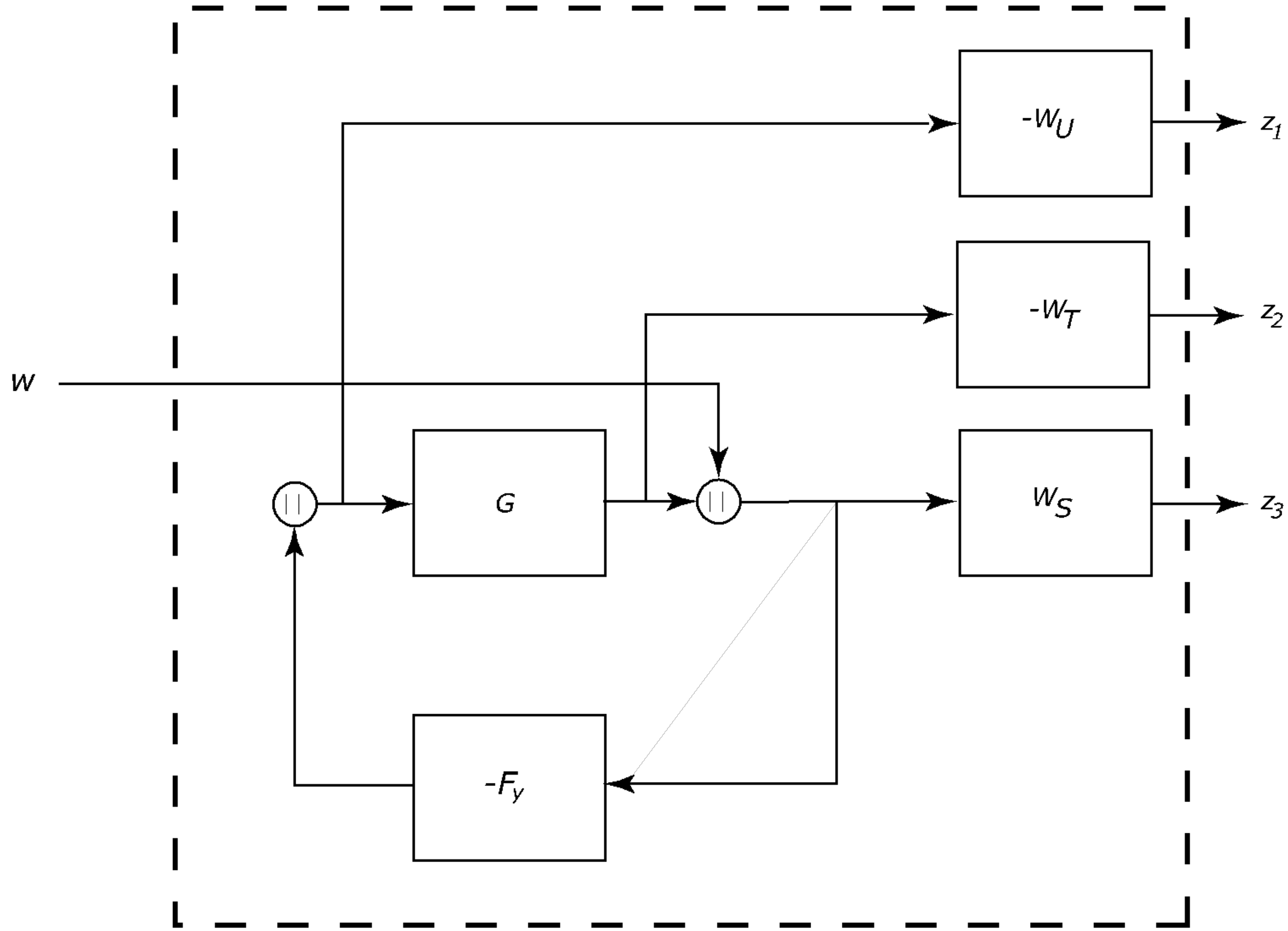
The extended system



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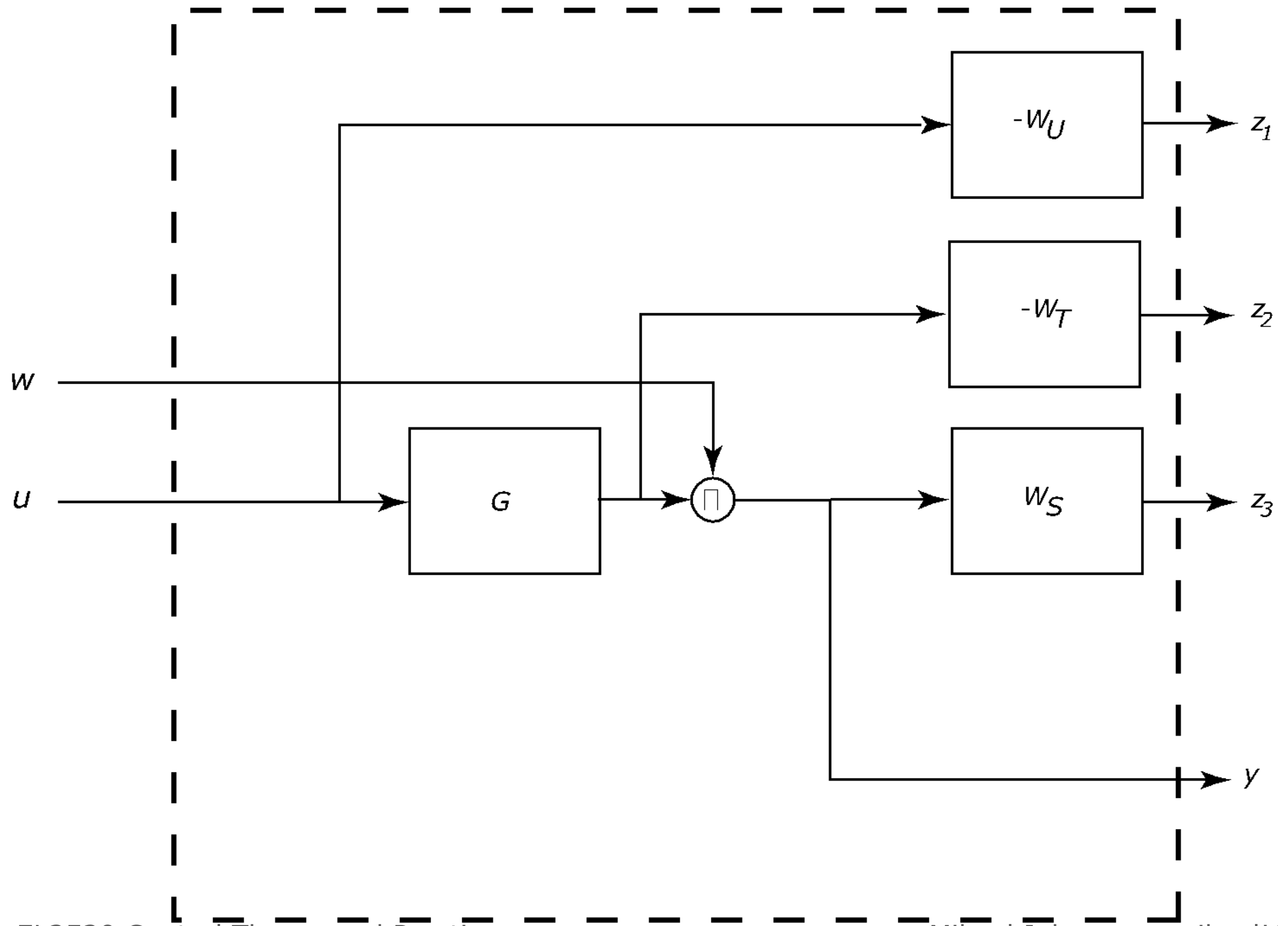
The extended system (cont'd)

Control design is based on the following model



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The control design problem

Find controller that ensures that gain from w→z is less than γ, i.e.

$$\int z(t)^T z(t)\, dt \leq \gamma^2 \int w(t)^T w(t)\, dt$$

Model:

$$\dot{x} = Ax + Bu + Nw$$
$$y = Cx + w$$
$$z = Mx + Du$$

Assumption:

$$D^T \begin{bmatrix} M & D \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$$

Note: Dynamics of weights W_S, W_T, W_U part of system dynamics.

H_∞ optimal control

Let P be a positive definite matrix solution to the Riccati equation

$$A^T P + PA + M^T M + P(\gamma^{-2} N N^T + B B^T) P = 0$$

If A-BB^{T}P is stable, the controller

$$\frac{d}{dt}\hat{x}(t) = A\hat{x} + Bu + N(y - C\hat{x})$$
$$u = -B^T P \hat{x}$$

fulfills the specifications (note: observer+state feedback)

Notes.

- controller order same as extended plant
- smallest gain can be found by a search over γ (e.g. by bisection)

How to select weights

Useful to constrain weights, limit number of ”tuning knobs” in design

1. Start with scalar weights (only use matrix-weights when needed). Make sure weights are stable and minimum phase (why?)

2. Use simple weights with easy interpretation.

   Ex. $W_S(s) = \dfrac{s/M_S + \omega_{BS}}{s + \omega_{BS} A}, \quad A << 1, \qquad W_U = 1$ or $W_U = \dfrac{s}{s + \omega_{BU}}$

3. Start shaping most important transfer matrix, then add one by one

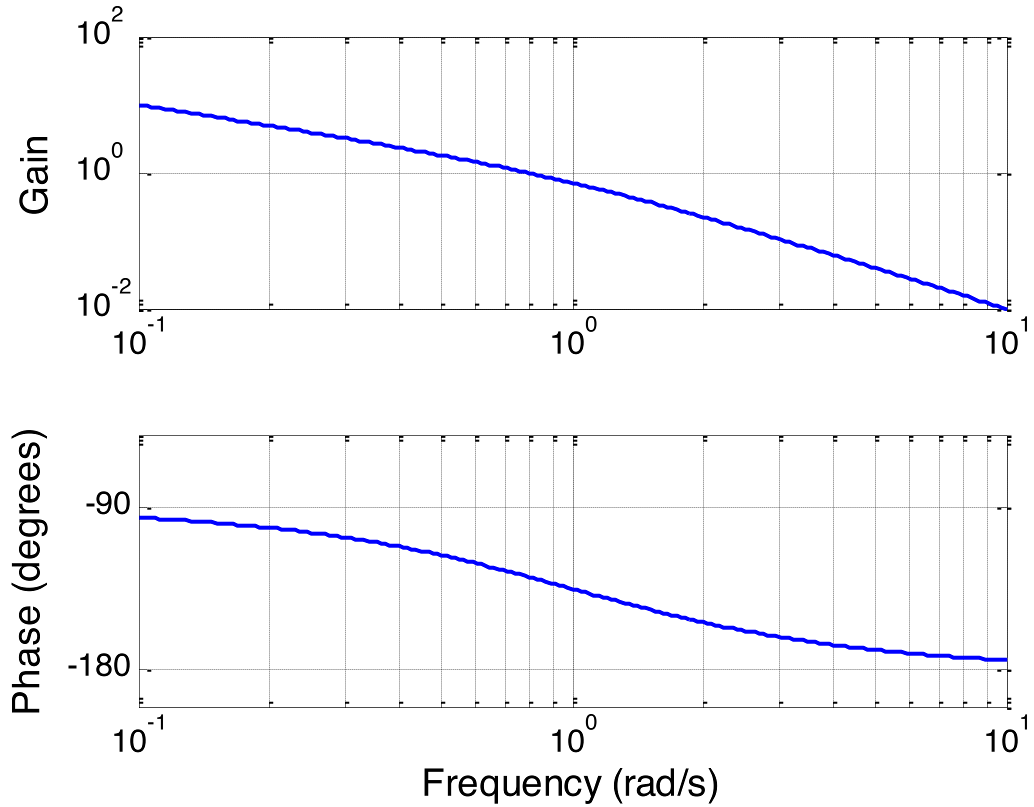
   Ex. first $\|W_S S\|_\infty$, then $\left\| \begin{pmatrix} W_S S \\ W_U U \end{pmatrix} \right\|_\infty$ then "full system"

4. When channels are very different, use diagonal weights

   $W_S(s) = \text{diag}\{W_{Si}(s)\}$

Example: DC servo

$$G(s) = \frac{1}{s(s+1)}$$



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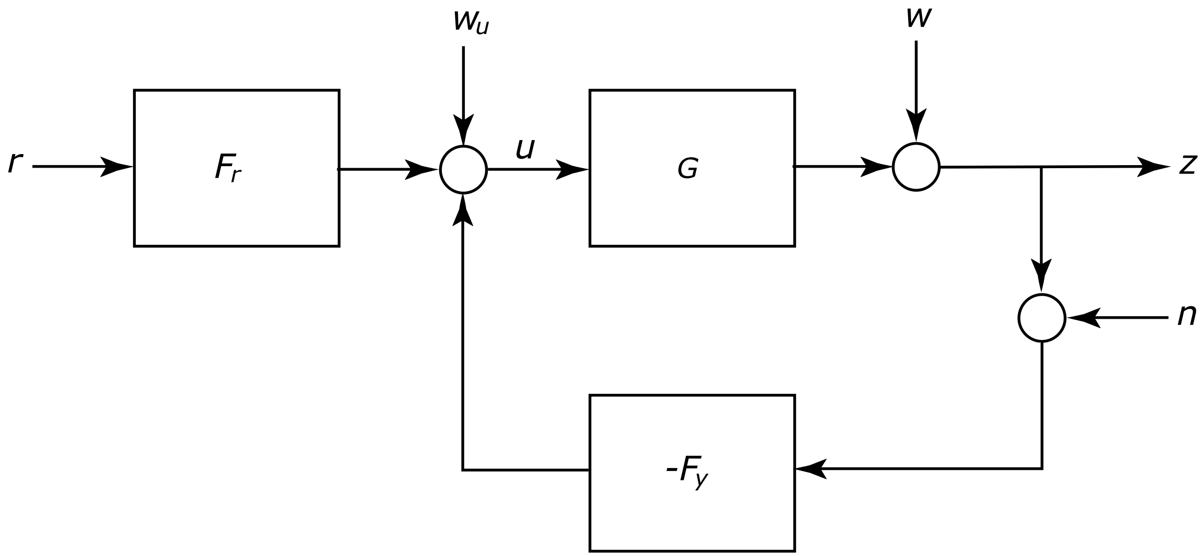
Specifications



Would like:

- Small influence of low-frequency (<0.01 rad/sec) disturbance w_u on output z. Maximum amplification 1.4 at any frequency.
- Limited amplification of high-frequency (>10 rad/sec) noise n in control signal u. Maximum amplification 1.4 at any frequency.
- Robust stability despite high-frequency uncertainty

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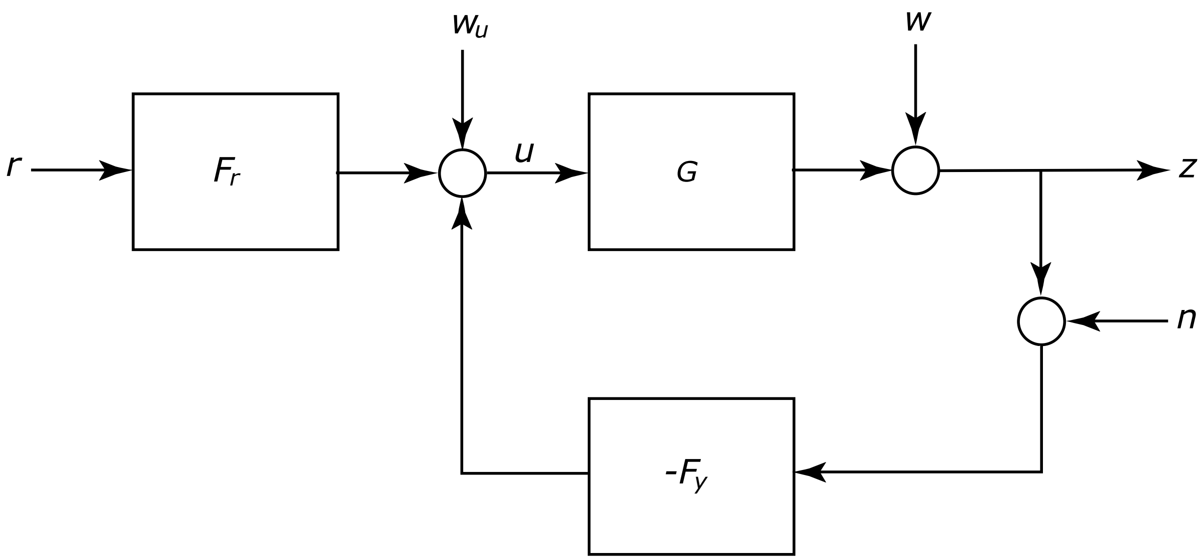
Quiz: What transfer functions?



- $w_u \rightarrow z$: SG
- Robust stability: T
- $n \rightarrow u$: SF_y

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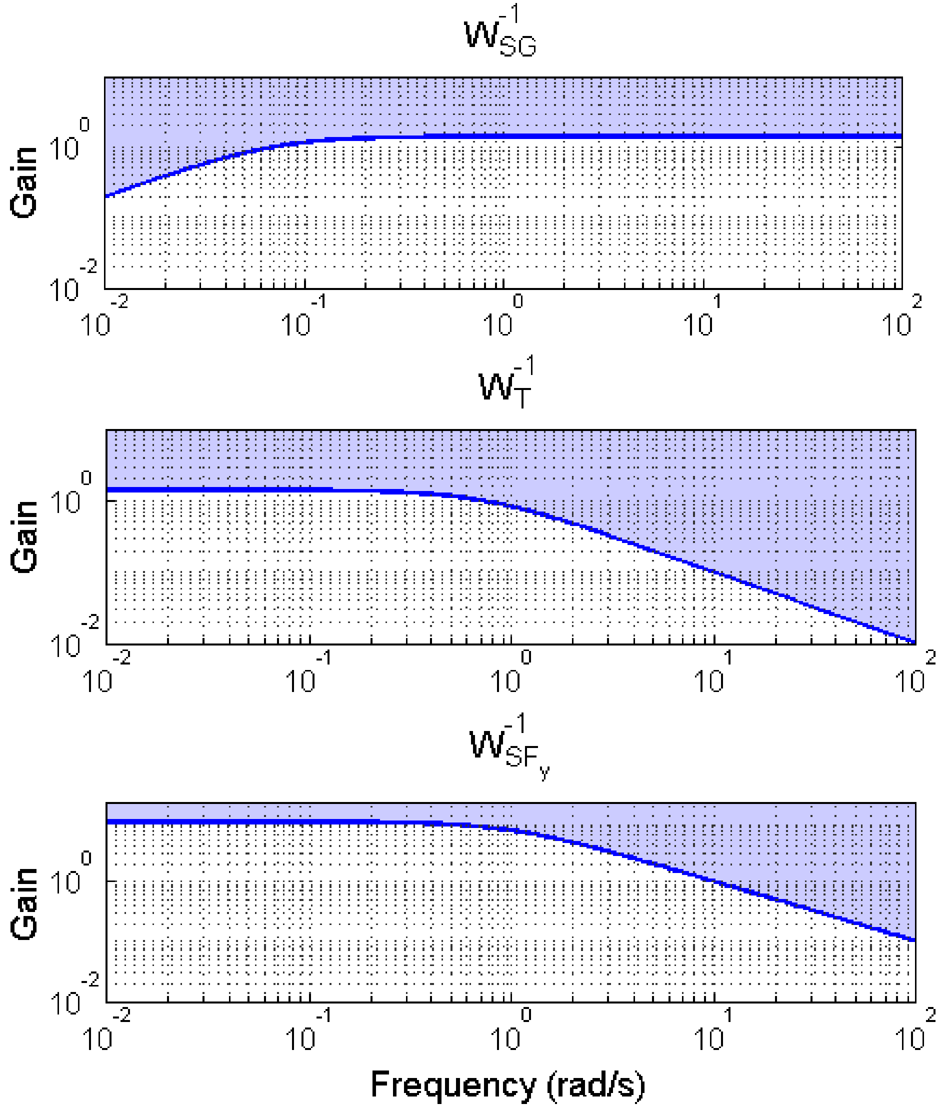
Specifications



$$\|W_{SG}SG\|_\infty \leq 1$$

$$\|W_T T\|_\infty \leq 1$$

$$\|W_{SF_y}SF_y\|_\infty \leq 1$$

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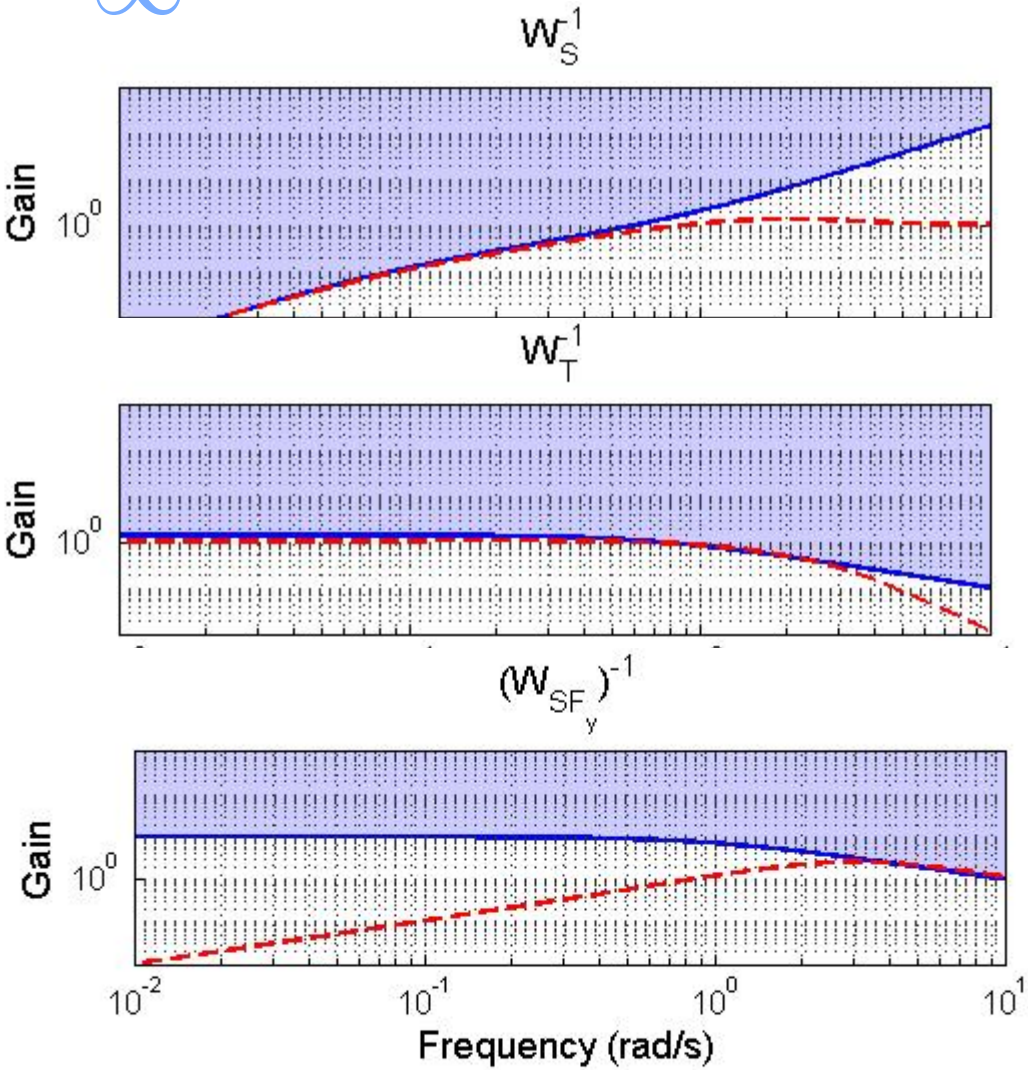
H_∞-optimal controller



Example

Consider the system with RHP zero

$$G(s) = \frac{1}{(0.2s+1)(s+1)} \begin{bmatrix} 1 & 1 \\ 1+2s & 2 \end{bmatrix}$$

RHP zero at z=0.5, with corresponding input direction (1,-1)

A first design…

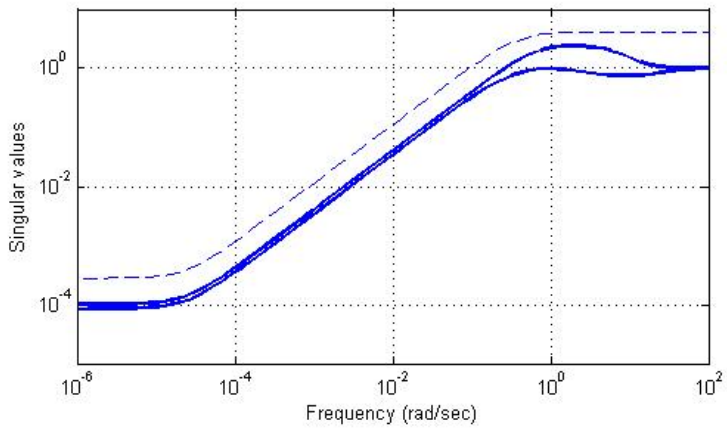
Since system has RHP zero at z=0.5, a reasonable weight is

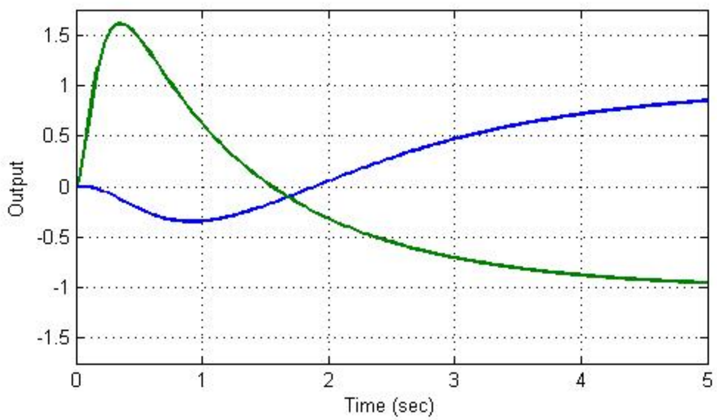
$$W_S(s) = \frac{s/M_S + \omega_{BS}}{s + \omega_{BS} A} \;\; M_S = 1.5,\; \omega_{BS} = 0.25,\; A = 1E-4$$

$$W_U(s) = 1$$

The mixed sensitivity design achieves $\gamma_{\min} = 2.79$

Reasonable sensitivities, but poor time-domain performance.





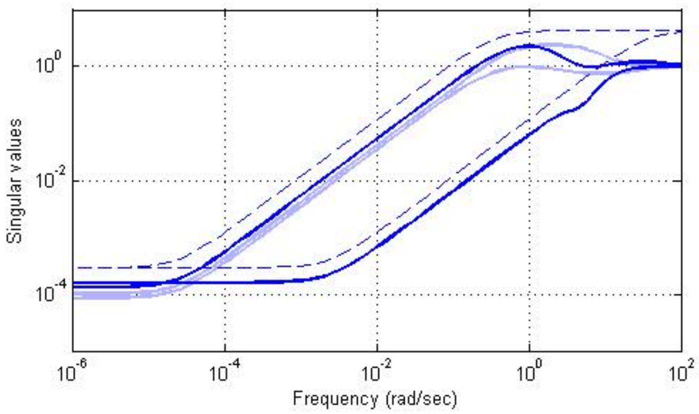
A second design…

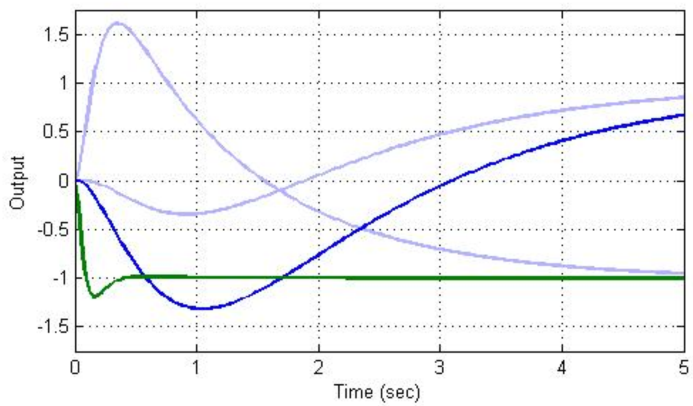
Can shift bandwidth limitation from one channel to the other
(i.e. alter singular vectors; limitation on maximum singular value remains)

$$W_S(s) = \text{diag}\{\frac{s/M_{Si} + \omega_{BSi}}{s + \omega_{BSi}Ai}\} \;\; M_{Si} = 1.5,\ \omega_{BS1} = 0.25,\ \omega_{BS2} = 25$$

$$W_U(s) = 1$$

Time response on channel two much better, constraint on bandwidth of maximum singular value of S still present.





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Conclusions

Modern loop shaping

- Mixed sensitivity design: minimizing H_∞ norm of extended system
- Optimal solution is state-feedback plus observer
- Tuning knobs for design are weight functions
- Weight selection: part art, part science (must practice!)

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