EL2520
Control Theory and Practice

Lecture 7:
Multivariable loop shaping

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Modern loop shaping

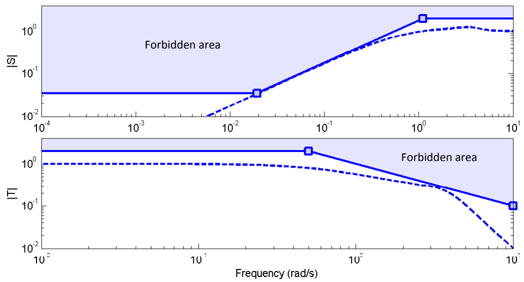
The modern view of control systems analysis and design
- signal norms, system gains, frequency responses
- constraints on many transfer functions (cf. "gang of six")
- fundamental limitations, robustness

Extends from SISO to MIMO

Today's lecture: modern loop shaping using H-infinity design
- Specify controller performance in terms of frequency responses
- Convert these constraints to weights
- Constrain norms of weighted transfer functions
- Directly design optimal controllers

Sensitivity shaping

Convenient to design controller by constraining critical transfer matrices



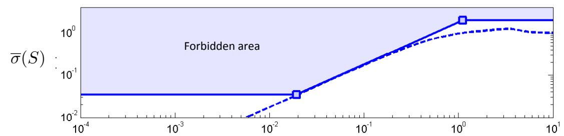
$$\|W_S S\|_\infty \leq 1$$
$$\|W_T T\|_\infty \leq 1$$
$$\vdots$$

Constraining singular values

If we assume that W_S is scalar, then $\|W_S S\|_\infty \leq 1$ is implied by

$$|W_S(i\omega)| \cdot \overline{\sigma}(S(i\omega)) \leq 1 \qquad \forall \omega$$

Hence, W_S defines a "forbidden area" for the singular values.



Fundamental limitations

As for SISO, non-minimum phase elements limit how we can select weights

Theorem. Let G have a zero z in the right half plane, and let the scalar transfer function W_S be stable and minimum phase. Then, a necessary condition for

$$\|W_S S\|_\infty = \sup_\omega \overline{\sigma}(W_S(i\omega)S(i\omega)) \leq 1$$

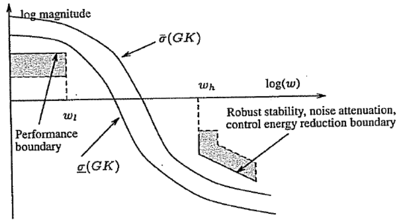
is that

$$|W_S(z)| \leq 1$$

Proof is analogous to SISO case (see book).

Multivariable loop shaping

As in SISO-case, constraints on S and T can be translated to constraints on the (**singular values** of the) loop gain L.

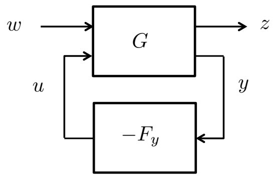


Hard to use for manual design of controllers (why?)

Optimization-based design

Sensitivity-shaping can be done using the following result:

Fact: It is possible to find a controller that ensures that the gain from w to z is less than γ whenever such a controller exists.



The controller is linear and of the same order as G.

Approximate design specifications

Problem: find controller that satisfies specifications

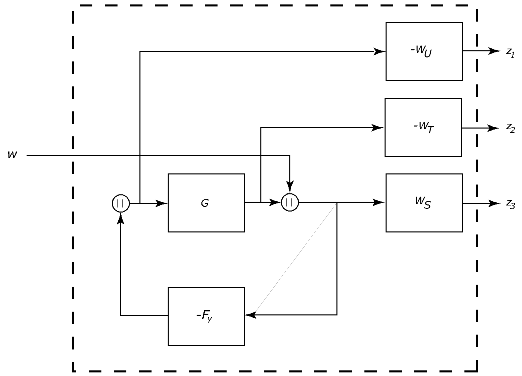
$$\|W_S S\| \leq 1$$
$$\|W_T T\| \leq 1$$
$$\|W_{SF_y} S F_y\| \leq 1$$

If we consider the approximate constraint

$$\left\| \begin{pmatrix} W_S S \\ W_T T \\ W_{SF_y} S F_y \end{pmatrix} \right\| \leq 1$$

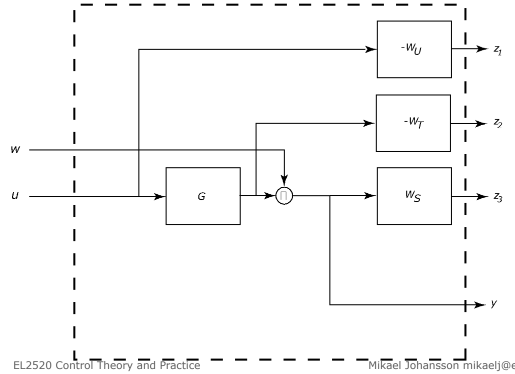
this can be viewed as the norm of an "extended system"

The extended system



The extended system (cont'd)

Control design is based on the following model



The control design problem

Find controller that ensures that gain from w→z is less than γ, i.e.

$$\int z(t)^T z(t)\,dt \leq \gamma^2 \int w(t)^T w(t)\,dt$$

Model:

$$\dot{x} = Ax + Bu + Nw$$
$$y = Cx + w$$
$$z = Mx + Du$$

Assumption:

$$D^T \begin{bmatrix} M & D \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$$

Note: Dynamics of weights W_S, W_T, W_U part of system dynamics.

H_∞ optimal control

Let P be a positive definite matrix solution to the Riccati equation

$$A^T P + PA + M^T M + P(\gamma^{-2} N N^T + BB^T)P = 0$$

If $A-BB^TP$ is stable, the controller

$$\frac{d}{dt}\hat{x}(t) = A\hat{x} + Bu + N(y - C\hat{x})$$
$$u = -B^T P \hat{x}$$

fulfills the specifications (note: observer+state feedback)

Notes.

- controller order same as extended plant
- smallest gain can be found by a search over γ (e.g. by bisection)

How to select weights

Useful to constrain weights, limit number of "tuning knobs" in design

1. Start with scalar weights (only use matrix-weights when needed). Make sure weights are stable and minimum phase (why?)
2. Use simple weights with easy interpretation.

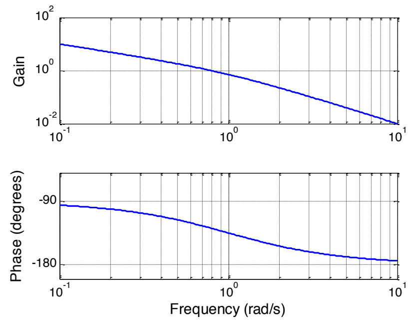
   Ex. $W_S(s) = \dfrac{s/M_S + \omega_{BS}}{s + \omega_{BS}A}$, $\quad A << 1$, $\qquad W_U = 1$ or $W_U = \dfrac{s}{s + \omega_{BU}}$
3. Start shaping most important transfer matrix, then add one by one

   Ex. first $\|W_S S\|_\infty$, then $\left\| \begin{pmatrix} W_S S \\ W_U U \end{pmatrix} \right\|_\infty$ then "full system"
4. When channels are very different, use diagonal weights

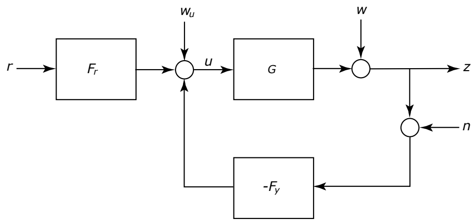
   $W_S(s) = \text{diag}\{W_{Si}(s)\}$

Example: DC servo

$$G(s) = \frac{1}{s(s+1)}$$



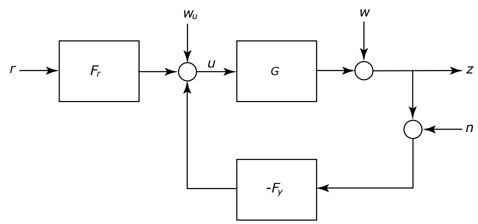
Specifications



Would like:

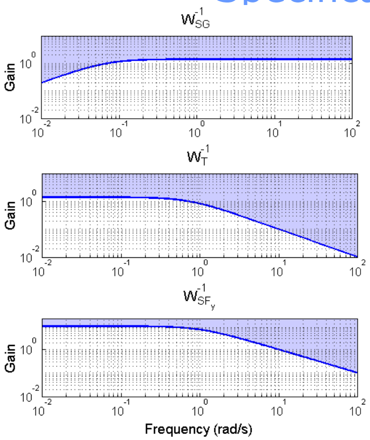
- Small influence of low-frequency (<0.01 rad/sec) disturbance w_u on output z. Maximum amplification 1.4 at any frequency.
- Limited amplification of high-frequency (>10 rad/sec) noise n in control signal u. Maximum amplification 1.4 at any frequency.
- Robust stability despite high-frequency uncertainty

Quiz: What transfer functions?



- $w_u \rightarrow z$: SG
- Robust stability: T
- $n \rightarrow u$: SF_y

Specifications

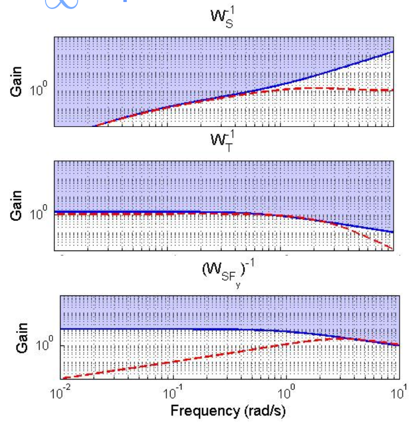


$$\|W_{SG}SG\|_\infty \leq 1$$

$$\|W_T T\|_\infty \leq 1$$

$$\|W_{SF_y}SF_y\|_\infty \leq 1$$

H_∞-optimal controller



Example

Consider the system with RHP zero

$$G(s) = \frac{1}{(0.2s+1)(s+1)}\begin{bmatrix} 1 & 1 \\ 1+2s & 2 \end{bmatrix}$$

RHP zero at z=0.5, with corresponding input direction (1,-1)

A first design...

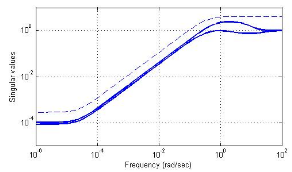
Since system has RHP zero at z=0.5, a reasonable weight is

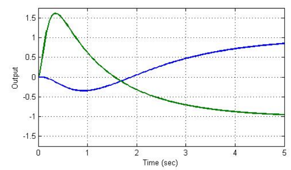
$$W_S(s) = \frac{s/M_S + \omega_{BS}}{s + \omega_{BS}A} \quad M_S = 1.5,\ \omega_{BS} = 0.25,\ A = 1E-4$$

$$W_U(s) = 1$$

The mixed sensitivity design achieves $\gamma_{\min} = 2.79$

Reasonable sensitivities, but poor time-domain performance.





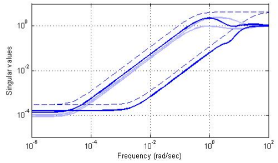
A second design…

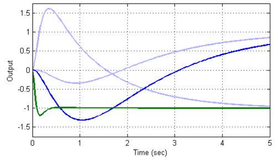
Can shift bandwidth limitation from one channel to the other
(i.e. alter singular vectors; limitation on maximum singular value remains)

$$W_S(s) = \text{diag}\{\frac{s/M_{Si} + \omega_{BSi}}{s + \omega_{BSi}Ai}\} \; M_{Si} = 1.5,\ \omega_{BS1} = 0.25,\ \omega_{BS2} = 25$$

$$W_U(s) = 1$$

Time response on channel two much better, constraint on bandwidth
of maximum singular value of S still present.





Conclusions

Modern loop shaping

- Mixed sensitivity design: minimizing H_∞ norm of extended system
- Optimal solution is state-feedback plus observer
- Tuning knobs for design are weight functions
- Weight selection: part art, part science (must practice!)