Repetition:

Decentralized control:

See the MIMO system as a bunch of SISO systems where you control each output with a separate input.

Pairing problem:

Which input controls which output?

$$RGA = G \times (G^{-1})^T$$

elementwise product

1, avoid negative elements in $RGA(0)$

2, Select a pairing with elements close to 1 in $RGA(i\omega_c)$

Decoupling:

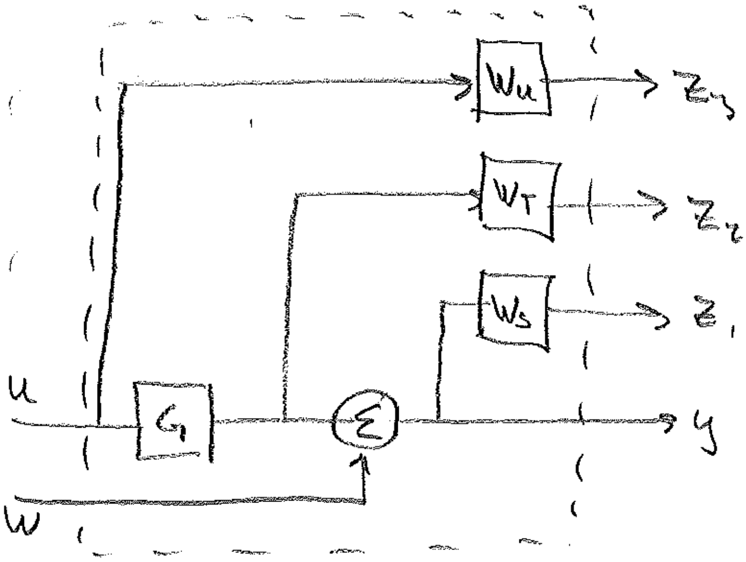
Make the loops more independent by adding "feed forward" to counter coupling-effects.

Theory H_∞:

Consider the extended system

$$\begin{pmatrix} z \\ y \end{pmatrix} = G_e(s) \begin{pmatrix} w \\ u \end{pmatrix}$$

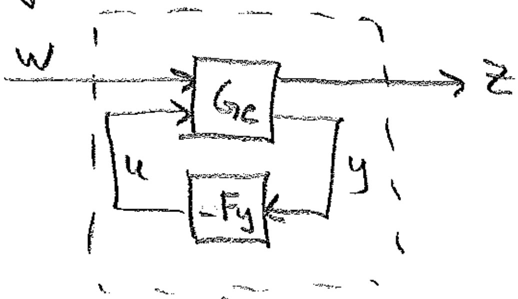
e.g. given by



G_e

By letting $u = -F_y y$ we get

$$Z(s) = G_{ec}(s) W(s) \quad \text{with} \quad G_{ec} = \begin{bmatrix} W_S S \\ -W_T T \\ W_u G_{wu} \end{bmatrix}$$



G_{ec}

H_∞-control: Objective $\|G_{ec}\|_\infty < \gamma$

By designing F_y such tha $\|G_{ec}\|_\infty \leq \gamma$ we ensure

$\|W_S S\| < \gamma$, $\|W_T T\|_\infty < \gamma$, $\|W_u G_{wu}\|_\infty < \gamma$

Where W_S , W_T & W_u are weights ensuring performance, robustness etc.

In state space G_e can be expressed as

$$\begin{cases} \dot{x} = Ax + Bu + Nw \\ z = Mx + Du \\ y = Cx + w \end{cases}$$

Where we assume that $D^T[M\ D] = [0\ I]$

An H_∞ controller ensuring $\|G_{ec}\|_\infty \leq \gamma$ is then

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + N[y - C\hat{x}] & \leftarrow \text{observer} \\ u = -L\hat{x} & \leftarrow \text{state feedback from observed states;} \end{cases}$$

$L = B^T P$

P is the positive definite solution to

$$A^T P + PA + M^T M + P\left(\frac{1}{\gamma^2} NN^T - BB^T\right)P = 0$$

The given controller might be unstable So we need to check that

$A - BB^T P$ is stable. ("A-matrix for the closed loop.")

$= A - BL$

10.8

Assume that we have the system

$$Z(s) = G(s)U(s) + W(s)$$
$$Y(s) = Z(s) + N(s)$$

which we will control using an H_∞ controller derived for the weights

$$W_S(s) = \frac{s + \omega_S}{s}$$

$$W_T(s) = \frac{s + \omega_T}{\omega_T} \cdot \frac{1}{1+\epsilon s} \qquad \epsilon << 1/\omega_T$$

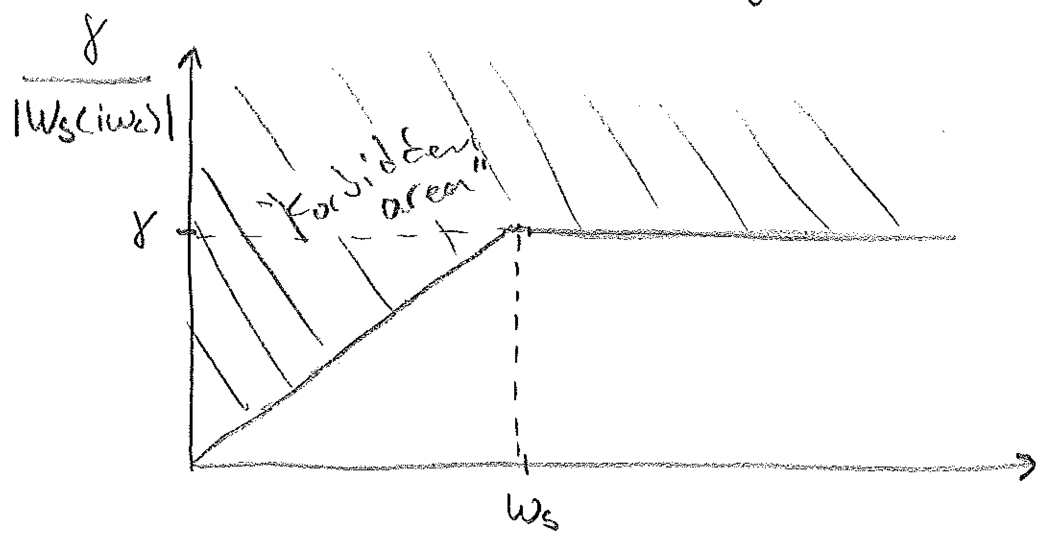
$$W_u(s) = 1$$

a) If we increase ω_S, should we expect better or worse damping of low frequency disturbances?

H_∞ control imply that we aim at getting

$$\|W_S S\|_\infty < \gamma \Rightarrow |S(i\omega)| < \frac{\gamma}{|W_S(i\omega)|} \quad \forall \omega$$

If we plot $\frac{\gamma}{W_S}$ asymptotically we get

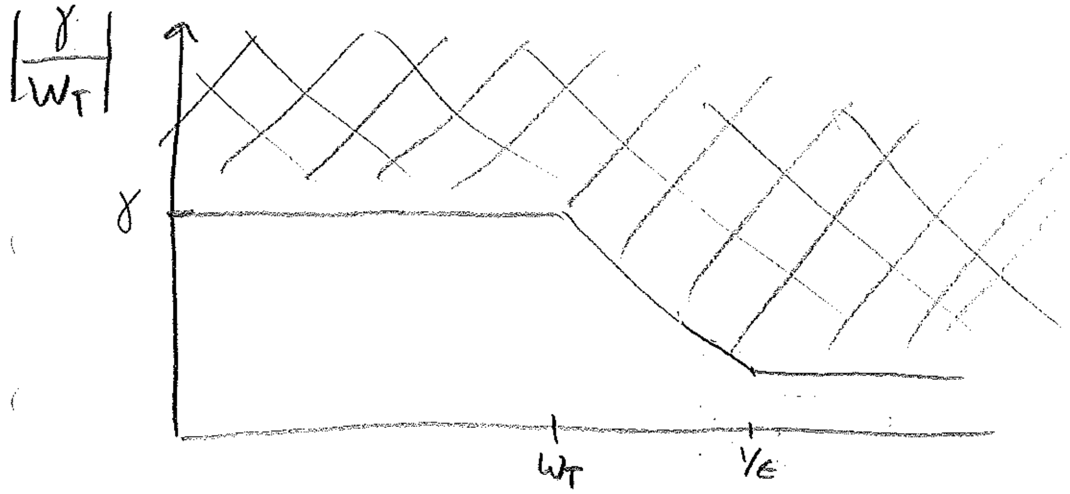


larger $\omega_S \Rightarrow$ better damping.

10.8

b) If we increase ω_T, should we expect a higher or lower bandwidth?

Same thing, we plot $\left|\frac{\gamma}{W_T}\right|$ asymptotically



ω_T larger $\Rightarrow$ higher bandwidth

Exercise 10.2:

Given the system

$$y = \frac{1}{1+s} u$$

and the constraints

$$|S(i\omega)| \leq \gamma\omega$$

$$|T(i\omega)| \leq 2\gamma \qquad \forall\omega > 0$$

$$|G_{wu}(i\omega)| \leq 0{,}2\gamma \qquad \gamma > 0$$

Write down the equations and conditions that determines a controller, fulfilling the constraints.

We start by expressing the constraints in terms of a weighted norm.

$$|S(i\omega)| < \gamma\omega \Leftarrow \frac{|S(i\omega)|}{|i\omega|} < \gamma$$

Let $W_S = \frac{1}{s}$. Then

$$\frac{|S(i\omega)|}{|i\omega|} = |W_S(i\omega) S(i\omega)|$$

We also have

$$|W_S(i\omega) S(i\omega)| \leq \sup_{\omega} |W_S(i\omega) S(i\omega)| = \|W_S S\|_\infty$$

Hence $\|W_S S\|_\infty < \gamma \Rightarrow |S(i\omega)| < \gamma\omega$

Similarily, let $W_T = \frac{1}{2}$ & $W_u = \frac{1}{0{,}2} = 5$

$\|W_T T\|_\infty < \gamma \Rightarrow |T(i\omega)| < \gamma$

$\|W_u G_{wu}\|_\infty < \gamma \Rightarrow |G_{wu}(i\omega)| < \gamma$

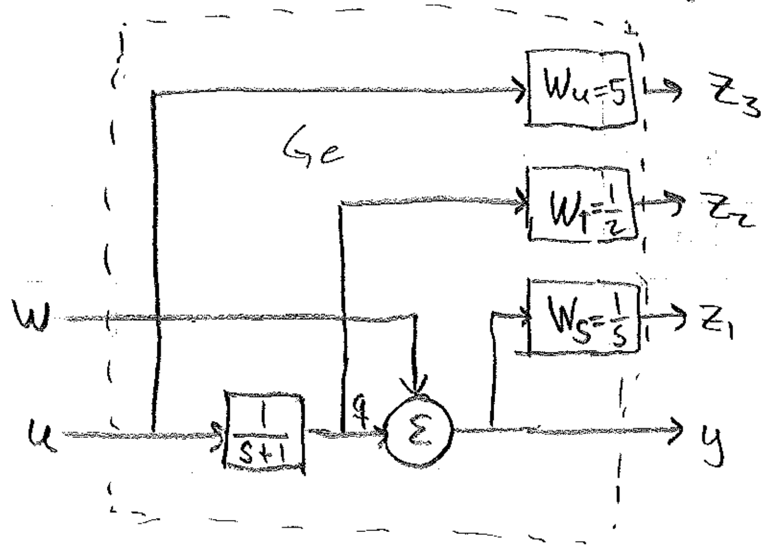
Since $\left\| \begin{matrix} G_1 \\ G_2 \end{matrix} \right\|_\infty \geq \max(\|G_1\|_\infty, \|G_2\|_\infty)$

it follows that

$$\left\| \begin{matrix} W_S S \\ -W_T T \\ W_u G_{wu} \end{matrix} \right\|_\infty < \gamma \Rightarrow \begin{matrix} |S(i\omega)| < \gamma\omega \\ |T(i\omega)| < 2\gamma \\ |G_{wu}(i\omega)| < 0{,}2\gamma \end{matrix}$$

Thus, we want to find an H_∞-controller fulfilling the above.

We introduce the extended system



We want to find a state-space representation.

We have:

$$Q(s) = \frac{1}{s+1} U(s)$$

$$Z_1(s) = \frac{1}{s} Y(s)$$

$$Z_2(s) = \frac{1}{2} Q(s)$$

$$Z_3(s) = 5 \quad U(s)$$

$$Y(s) = Q(s) + W(s)$$

We take the inverse laplace transform

$$\Rightarrow \quad \dot{q} = -q + u$$

$$\dot{z}_1 = y \qquad \Rightarrow \dot{z}_1 = q + w$$

$$z_2 = \frac{1}{2} q$$

$$z_3 = 5 \; u$$

$$y = q + w$$

Let $x = \begin{pmatrix} q \\ z_1 \end{pmatrix}$ then we get

$$\dot{x} = \underbrace{\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{B} u + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{N} w$$

$$z = \underbrace{\begin{pmatrix} 0 & 1 \\ 1/2 & 0 \\ 0 & 0 \end{pmatrix}}_{M} x + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}}_{D} u$$

State space on standard form (10.7 in Swedish book)

$$y = \underbrace{(1 \;\; 0)}_{C} x + w$$

Since $D^T[M\ D] = [0\ 0\ 25] \neq [0\ 0\ 1]$ we need to make a change of variables.

We note that for $\tilde{D} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ we get $\tilde{D}^T[M\ \tilde{D}] = [0\ 0\ 1]$ which lead us to the variable change $\tilde{u} = 5u$ since then

$\tilde{D}\tilde{u} = Du$.

We also need $\tilde{B}\tilde{u} = Bu \Rightarrow \tilde{B} = B \cdot \frac{1}{5} = \begin{bmatrix} \frac{1}{5} \\ 0 \end{bmatrix}$

Hence, the system can be written as

$$\dot{x} = Ax + \tilde{B}\tilde{u} + Nw$$

$$z = Mx + \tilde{D}\tilde{u}$$
$$y = Cx + w$$

The H_∞ optimal controller is then given by

$$\dot{\hat{x}} = A\hat{x} + \tilde{B}\tilde{u} + N[y - C\hat{x}]$$
$$\tilde{u} = -L_\infty \hat{x}$$

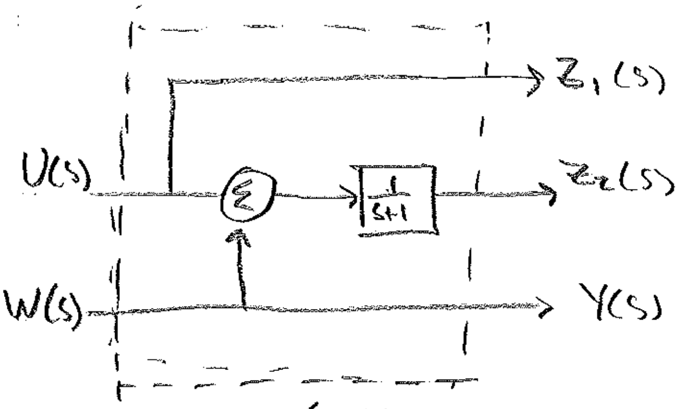
Where $L_\infty = \tilde{B}^T P$

and P is the positive definite solution to

$$A^T P + PA + M^T M + P(\gamma^{-2} NN^T - \tilde{B}\tilde{B}^T)P = 0$$

Such that $A - \tilde{B}\tilde{B}^T P$ is stable.

10.4 Consider the extended system



$G_0(s)$

a) Show that

$$\begin{cases} \dot{x} = -x + u + w \\ z = \begin{pmatrix} 0 \\ 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \\ y = w \end{cases}$$

is a state space description of $G_0(s)$

What is $G_0(s)$? 3 outputs, 2 inputs $\Rightarrow$ 3×2 matrix.

$$\begin{pmatrix} Z_1(s) \\ Z_2(s) \\ Y(s) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ \frac{1}{s+1} & \frac{1}{s+1} \\ 0 & 1 \end{pmatrix}}_{G_0(s)} \begin{pmatrix} U(s) \\ W(s) \end{pmatrix}$$

We rewrite the state space model as

$$\dot{x} = \overbrace{-1}^{A} \cdot x + \overbrace{[1 \quad 1]}^{B} \begin{pmatrix} u \\ w \end{pmatrix}$$

$$\begin{pmatrix} z_1 \\ z_2 \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{C} x + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}}_{D} \begin{pmatrix} u \\ w \end{pmatrix}$$

$$G_0(s) = C(sI - A)^{-1}B + D = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (s+1)^{-1} \begin{pmatrix} 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 \\ \frac{1}{s+1} & \frac{1}{s+1} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{s+1} & \frac{1}{s+1} \\ 0 & 1 \end{pmatrix}$$

10.4 b We are given a control structure

$$\begin{cases} \dot{\hat{x}} = -\hat{x} + u + y \\ u = -L\hat{x} \end{cases}$$

Determine L such that $\|G_{ec}\|_\infty < 1 = \gamma$

This looks like an H_∞-controller!

We check that the structure actually match.

The state-space model in a) can be expressed as

$$\dot{x} = Ax + Bu + Nw \qquad\qquad A = -1 \quad B = 1 \quad N = 1$$
$$z = Mx + Du \qquad \text{with} \quad M = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$y = Cx + w \qquad\qquad C = 0$$

We note that $D^T[M\ D] = [0\ 1]$.

The H_∞-controller should be

$$\dot{\hat{x}} = A\hat{x} + Bu + N[y - C\hat{x}] = -\hat{x} + u + y$$
$$u = -L\hat{x}$$

Which match the given structure. Hence with $\gamma = 1$ we have

$L = BP$ wher P is given by

$$A^T P + PA + M^T M + P\left(\frac{1}{\gamma^2} NN^T - BB^T\right) P = 0$$

$$-P - P + \underbrace{[0\ 1]\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{1} + P(1 \cdot 1 \cdot 1 - 1 \cdot 1)P = 0$$

$$-2P + 1 = 0 \Rightarrow P = 1/2 \quad \text{(positive definite!)}$$

$L = B^T P = 1 \cdot \frac{1}{2} = \frac{1}{2}.$

Is the controller stable? The closed loop poles are given by the eigenvalues of $(A - BL)$. $A - BL = -1 - \frac{1}{2} = -\frac{3}{2}$ (scalar!)

Hence the only pole is $s = \frac{-3}{2}$ $\Rightarrow$ Closed loop stable.

c) Determine the L that minimize $\|G_{ee}\|_\infty$ and compute the transfer function for that controller.

(I) $-2P + 1 + P^2\left(\frac{1}{\gamma^2} - 1\right) = 0$ must have a positive real solution.

(II) $A - BL$ must be stable.

(I) We have a quadratic equation: For $\gamma^2 \neq 1$ we get

$$P^2 - \frac{2\gamma^2}{1-\gamma^2} P + \frac{\gamma^2}{1-\gamma^2}$$

$\Rightarrow$

$$P = \frac{\gamma^2}{1-\gamma^2} \pm \sqrt{\left(\frac{\gamma^2}{1-\gamma^2}\right)^2 - \frac{\gamma^2}{1-\gamma^2}} = \frac{\gamma^2}{1-\gamma^2} \pm \frac{\gamma \cdot \sqrt{2\gamma^2 - 1}}{|1-\gamma^2|}$$

For this to have a real solution we need

$\gamma \geq \frac{1}{\sqrt{2}}$

If we insert $\gamma = 1/\sqrt{2}$ we get

$$P = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1 \Rightarrow L = B^T P = 1 \cdot 1 = 1$$

(II) $A - BL = -1 - 1 = -2 \Rightarrow$ pole at -2.

In conclusion $\min \|G_{ec}\|_\infty = \frac{1}{\sqrt{2}}$

which is achieved for $L = 1$

So what is the transfer function of this controller?

$$\dot{\hat{x}} = -\hat{x} + Bu + y$$
$$u = -L\hat{x}$$

We take the Laplace transform $\Rightarrow$

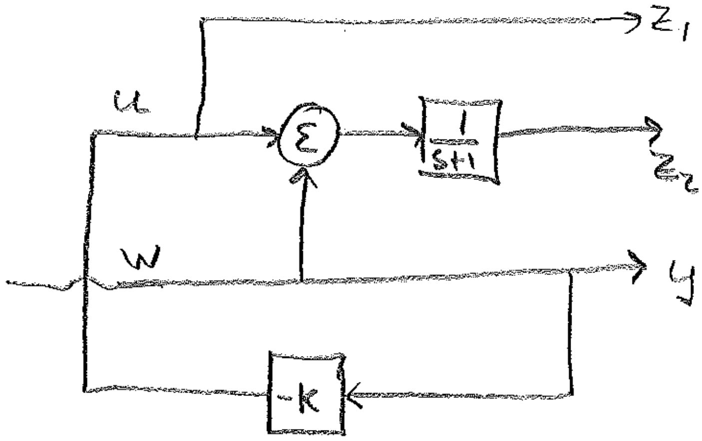
$$\begin{cases} s\hat{X}(s) = -\hat{X}(s) + BU(s) + Y(s) \\ U(s) = -L\hat{X}(s) \Rightarrow \hat{X}(s) = -U(s) \end{cases}$$

$$\Rightarrow \quad -sU(s) = U(s) + U(s) + Y(s) \Rightarrow$$

$$U(s) = \underbrace{\frac{-1}{s+2}}_{-F_y} Y(s)$$

d) Assume we use proportional control $U(s) = -KY(s)$. Which K minimize $\|G_{ec}\|_\infty$?

We have the system



What is G_{ec}?

$$\begin{pmatrix} Z_1(s) \\ Z_2(s) \end{pmatrix} = \underbrace{\begin{pmatrix} -K \\ \frac{1}{s+1}(-K+1) \end{pmatrix}}_{G_{ec}} W(s)$$

$$\|G_{ec}\|_\infty = \sup_\omega \bar{\sigma}(G_{ec}(i\omega))$$

What is $\bar{\sigma}$? Square root of eigenvalues of

$$G_{ec}(i\omega)^* G_{ec}(i\omega) = \begin{pmatrix} -K & \frac{-K+1}{-i\omega+1} \end{pmatrix} \begin{pmatrix} -K \\ \frac{-K+1}{i\omega+1} \end{pmatrix} =$$

$$= K^2 + \frac{(-K+1)^2}{1+\omega^2}$$ (a scalar, i.e it is its own eigenvalues).

$$\Rightarrow \quad G^2 = K^2 + \frac{(1-K)^2}{1+\omega^2}$$

This is clearly maximized by $\omega = 0 \Rightarrow$

$$\|G_{ec}\|_\infty^2 = K^2 + (1-K)^2 = 2K^2 - 2K + 1$$

(note that the same K minimize $\|G_{ec}\|_\infty^2$ & $\|G_{ec}\|_\infty$)

Which is the minimizing K? We take the derivative

$$\frac{d\|G_{ec}\|_\infty^2}{dK} = 4K - 2 = 0 \Rightarrow K = \frac{1}{2}$$

$$\Rightarrow \|G_{ec}\|_\infty^2 = \frac{1}{2} \Rightarrow \|G_{ec}\|_\infty = \frac{1}{\sqrt{2}}$$