EL2520
Control Theory and Practice

Lecture 8:
Linear quadratic control

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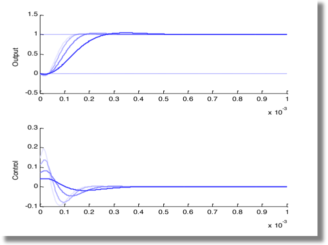
Linear quadratic control

Allows to compute the controller $F_y(s)$ that minimizes

$$J = \lim_{T\to\infty} \frac{1}{T}\int_0^T [z^T Q_1 z + u^T Q_2 u]\, dt$$

for given (positive definite) weight matrices Q_1 and Q_2.

Easy to influence control energy/transient performance trade-off



Linear quadratic control

Disadvantage of linear-quadratic control:

- Robustness and frequency-domain properties only indirectly

Challenge: framework developed for stochastic disturbances

$$J = \mathbf{E}\left\{\lim_{T\to\infty} \frac{1}{T}\int_0^T [z^T Q_1 z + u^T Q_2 u]\, dt\right\}$$

- Need to review stochastic disturbance models
- Have to skip some details
  (continuous-time stochastic processes technically tricky)

Relation to H_∞ control will be explored in next lecture.

Learning aims

After this lecture, you should be able to

- model disturbances in terms of their spectra
- use spectral factorization to re-write disturbances as filtered white noise
- compute the LQG-optimal controller (observer/controller gains)
- describe how the LQG weights qualitatively affect the time responses

Material: course book 5.1-5.4 + 9.1-9.3 + 9.A

State-space form

State space description of multivariable linear system

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

where $x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p$

- x(t) is called the *state vector*,
- systems on state-space form often written as (A,B,C,D)

Transfer matrix computed as

$$G(s) = C(sI - A)^{-1}B + D$$

Controllability

The *state* $\tilde{x}$ is *controllable* if, given x(0)=0, there exists u(t) such that x(t)= $\tilde{x}$ for some t<∞.

The *system* is *controllable* if all $\tilde{x}$ are controllable.

The *controllability matrix*

$$S(A,B) = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times mn}$$

- Controllable states $\tilde{x}$ can be written as $\tilde{x} = S(A,B)\alpha$ for some $\alpha \in$ R^{mn}
- System is controllable if S(A,B) has full rank (i.e., for each x there exists α such that x=S(A,B)α)

Observability

The *state* $\tilde{x} \neq 0$ is *unobservable* if $x(0) = \tilde{x}$ and u(t)=0 for t>0 implies that y(t)=0 for t$\geq$ 0.
The *system* is *observable* if no states are unobservable

The *observability matrix*

$$O(A,C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{pm \times n}$$

- Unobservable states $\tilde{x}$ are solutions to $O(A,C)\tilde{x} = 0$
- System is observable if O(A,C) has full rank (i.e., only $\tilde{x} = 0$ solves $O(A,C)\tilde{x} = 0$)

Modifying dynamics via state feedback

Open loop system

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$

can be controlled using state feedback u(t)=-Lx(t) ➔

$$\frac{d}{dt}x(t) = (A - BL)x(t)$$

Q: can we choose L so that A-BL gets arbitrary eigenvalues?
A: if and only if the system is controllable.

Observers

The state vector of the system

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

can be estimated using an observer

$$\frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}) \Rightarrow$$
$$\frac{d}{dt}\tilde{x}(t) = (A - KC)\tilde{x}(t) \quad \text{where } \tilde{x}(t) = x(t) - \hat{x}(t)$$

Q: can we choose K so A-KC gets arbitrary eigenvalues?
A: if and only if system is observable

Stabilizability and detectability

Control objective concerns only outputs z of system, i.e., controllability and observability of all states may not be so important.

Exception: must be able to control and observe unstable modes!

A system (A,B) is *stabilizable* if there is L so that A-BL is stable

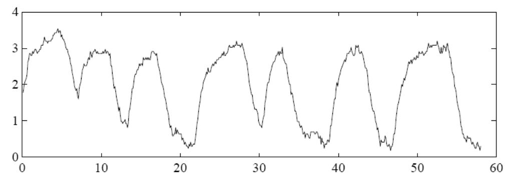
A system (A,C) is *detectable* if there is K so that A-KC is stable

Today's lecture

- Recap: State-space represenation, state feedback and observers
- Review: Modeling disturbances as filtered white noise
- Linear quadratic Gaussian (LQG) control

Disturbances

Disturbances model a wide range of phenomena that are not easily described in more detail, e.g.
- load variations, measurement noise, process variations, …



Important to model
- Size, frequency content and correlations between disturbances.

Signal sizes

So far in the course, we have used the 2-norm

$$\|z\|_2^2 = \int_0^\infty |z(t)|^2 \, dt$$

If the integral does not converge, we can use

$$\|z\|_e^2 = \lim_{T\to\infty} \frac{1}{T} \int_{t=0}^T |z(t)|^2 \, dt$$

A crude measure "size" (disregards frequency content of signal)

More informative measure

How are z(t) related to z(t-τ)? One measure is

$$r_z(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T z(t)z(t-\tau) \, dt$$

For ergodic stochastic processes, we have

$$r_z(\tau) = \mathbf{E} z(t) z(t-\tau)$$

(i.e., the covariance function of the signal)

Vector valued signals

For vector-valued z, coupling between z_i and z_j can be described by

$$r_{ij}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T z_i(t) z_j(t-\tau) \, dt$$

Can be combined into matrix

$$R_z(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T z(t) z^T(t-\tau) \, dt$$

For ergodic stochastic processes, we get

$$R_z(\tau) = \mathbf{E}\, z(t) z^T(t-\tau)$$

Signal spectra

Translating the signal measure to the frequency domain

$$\Phi_z(\omega) = \int_{-\infty}^{\infty} R_z(\tau) e^{i\omega\tau} \, d\tau$$

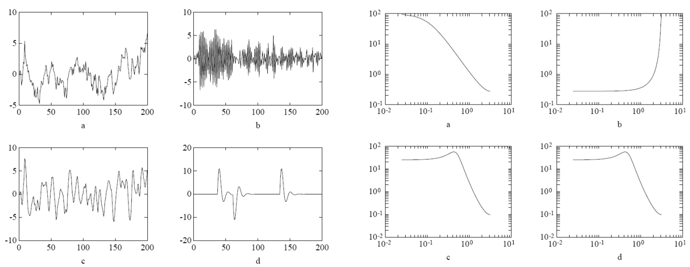
$\Phi_z(\omega)$ is called the *spectrum* of z.

Interpretation

- $[\Phi_z(\omega)]_{ii}$ measures the energy content of z_i at frequency ω
- $[\Phi_z(\omega)]_{ij}$ measures coupling of z_i and z_j at frequency ω
- $[\Phi_z(\omega)]_{ij} = 0$ implies that z_i and z_j are uncorrelated

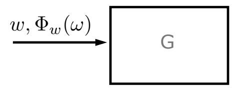
A signal with Φ_z constant for all ω is called *white noise,*
(in this case, we call Φ_z the *covariance matrix* of z)

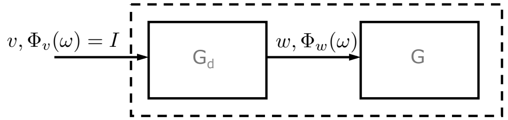
Examples: signals and spectra



Disturbances as filtered white noise

Fact (spectral factorization): any spectrum $\Phi(\omega)$ which is rational in ω^2, can be represented as white noise filtered through a stable non-minimum phase linear system.





(see course book Theorem 5.1 for a precise statement)

State-space model with distubances

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) + Nw_1(t)$$
$$y(t) = Cx(t) + Du(t) + w_2(t)$$

If disturbances w_1 and w_2 are not white, but have spectra that can be obtained via $w_i = G_i v_i$ where v_i is white noise, then we can re-write system as

$$\frac{d}{dt}\overline{x}(t) = \overline{A}\overline{x}(t) + \overline{B}u(t) + \overline{N}v_1(t)$$
$$y(t) = \overline{C}\overline{x}(t) + Du(t) + v_2(t)$$

Note: $\overline{x}$ is x augmented with the states from G_1, G_2;
$\overline{A}, \overline{B}, \ldots$ are A, B,… augmented with state-space descriptions of G_i

Today's lecture

- Recap: State-space represenation, state feedback and observers
- Recap: Modeling disturbances as filtered white noise
- Linear quadratic Gaussian (LQG) control

Linear quadratic Gaussian control

Model: linear system with white noise

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) + Nv_1(t)$$
$$y(t) = Cx(t) + v_2(t)$$
$$z(t) = Mx(t)$$

where v_1, v_2 are white noise with

$$\text{cov}([v_1,\ v_2]) = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$$

Objective: minimize effect of v on z, punish control cost

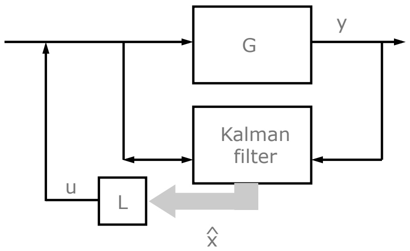
$$J = \mathbf{E}\left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T [z^T Q_1 z + u^T Q_2 u]\ dt \right\}$$

LQG: Linear system, Quadratic cost, Gaussian noise

Solution structure

Optimal solution satisfies *separation principle*, composed of

- Optimal linear state feedback (Linear-quadratic regulator)
- Optimal observer (Kalman filter)



Optimal solution

State feedback

$$u(t) = -L\hat{x}(t) = -Q_2^{-1}B^T S\hat{x}(t)$$

where S is the solution to the algebraic Riccati equation

$$A^T S + SA + M^T Q_1 M - SBQ_2^{-1}B^T S = 0$$

Kalman filter

$$\dot{\hat{x}}(t) = Ax(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

where $K=(PC^T+NR_{12})R_2^{-1}$ and P is the solution to

$$AP + PA^T + NR_1N^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T = 0$$

Example: LQR for scalar system

Scalar linear system

$$\dot{x}(t) = ax(t) + u(t), \qquad y(t) = x(t)$$

with cost

$$J = \int_0^\infty [x^2 + \rho u^2]\ dt$$

Riccati equation

$$2as + 1 - s^2/\rho = 0$$

has solutions

$$s = a\rho \pm \sqrt{(\rho a)^2 + \rho}$$

so the optimal feedback law is

$$u = -(s/\rho)x = -(a + \sqrt{a^2 + 1/\rho})x$$

Example: LQR for scalar system

Closed loop system

$$\dot{x}(t) = -(\sqrt{a^2 + 1/\rho})x(t)$$

Note: if system is unstable (a>0), then

- if control is expensive $\rho \to \infty$ then the minimum control input to stabilize the plant is obtained with the input u=-2|a|x, which moves the unstable pole to its mirror image –a
- if control is cheap ($\rho \to 0$), the closed loop bandwidth is roughly $1/\sqrt{\rho}$

Example: Scalar system Kalman filter

Scalar linear system

$$\dot{x}(t) = ax(t) + u(t) + v_1(t), \qquad y(t) = x(t) + v_2(t)$$

with covariances $E\{v_1^2\}=R_1$, $E\{v_2^2\}=R_2$, $E\{v_1v_2\}=0$.

Riccati equation

$$2ap + r_1 - p^2/r_2 = 0$$

gives

$$k = a + \sqrt{a^2 + r_1/r_2}$$

and estimation error dynamics

$$\frac{d}{dt}\tilde{x}(t) = -(\sqrt{a^2 + r_1/r_2})\tilde{x}$$

Interpretation: measurements discarded if too noisy.

The Servo Problem

Preferred way: augment system with reference model

$$\frac{d}{dt}x_{\text{ref}}(t) = A_{\text{ref}}x_{\text{ref}}(t) + Br_z(t)$$

$$e(t) = x_{\text{ref}}(t) - x(t)$$

where the reference model states are measurable, and use

$$J = \mathbf{E}\left\{\lim_{T\to\infty} \int_0^T [e^T Q_1 e + u^T Q_2 u]\, dt\right\}$$

However, when r is assumed to be constant the solution is

$$u(t) = -L\hat{x}(t) + L_{\text{ref}}r_z(t)$$

where L_{ref} is determined so that static gain of closed-loop (r_z→z) equals the identity matrix

LQG and loop shaping

LQG: simple to trade-off response-time vs. control effort
- but what about sensitivity and robustness?

These aspects can be accounted for using the noise models
- Sensitivity function: transfer matrix w_u→z
- Complementary: transfer matrix n→z

Example: S forced to be small at low frequencies by letting (some component of) w_1 affect the output of the system, and let w_1 have large energy at low frequencies,

$$w_1(t) = \frac{1}{p+\delta}v_1(t)$$

(delta small, strictly positive, to ensure stabilizability)

LQG Control: pros and cons

Pros:

- Simple to trade off response time vs. control effort
- Applies to multivariable systems

Cons:

- Often hard to see connection between weight matrices Q_1, Q_2, R_1, R_2 and desired system properties (e.g. sensitivity, robustness, etc)
- In practice, iterative process in which Q_1 and Q_2 are adjusted until closed loop system behaves as desired
- Poor robustness properties in general

Summary

- State-space theory recap:
  - Controllability, observability, stabilizability, detectability
  - State feedback and observers
- Modeling disturbances as white noise
  - Mean, covariance, spectrum
  - Spectral factorization: disturbances as filtered white noise
- Linear-quadratic controller
  - Kalman-filter + state feedback
  - Obtained by solving Riccati equations
  - Focuses on time-responses
  - Loop shaping and robustness less direct