Chapter 5

Rotational model

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5.1 Rotational model

So far we have considered the nucleus as a Fermi gas consisting of neutrons and protons moving freely under the influence of the shell model central potential. We have indicated that the success of the shell model was in the beginning considered a wonder since the very short range of the nuclear force acting upon the tightly packed nucleons make it difficult to understand their free motion. There are arguments to justify this, but still one would rather think that those characteristics are more suitable to generate a collective behavior where all nucleons move together as a whole preserving the nuclear volume. These considerations gave rise to the so-called liquid drop model in which the individual nucleons do not play any role. It is the nucleus as a whole that determines its dynamics. Like a liquid drop, the surface of the nucleus vibrates and rotates generating bands of quantized states. An even more limited collective model considers the nucleus as a rigid body with fixed center of mass. The only possible motions of such an object are rotations. This is the rotational model, where the conglomerate of individual nucleons form a compact entity, like a top or a rugby ball.

From classical mechanics it is known that three angles are needed to define the position of a rigid body with fixed center of mass. These are called Euler angles. One chooses an intrinsic axis system, called (x',y',z') in Fig. 5.1, and determines the direction of one of the axis, say z', with respect to the laboratory system (x,y,z). This operation requires two of the Euler angles. The third one is used to obtain the orientation of the body along the intrinsic axis z'. The Euler angles are denoted by (θ, ϕ, φ). The energy of the rotating rigid body, with the center of mass fixed at the center of coordinates, is

$$E = \frac{J_{x'}^2}{2\Im_{x'}} + \frac{J_{y'}^2}{2\Im_{y'}} + \frac{J_{z'}^2}{2\Im_{z'}} \tag{5.1}$$

where $\Im_{x'}$ is the x' component of the moment of inertia and $J_{x'}$ is the corresponding angular momentum component.

We will assume that the rigid body has cylindrical symmetry along the z' axis. Therefore the component $J_{z'}$ of the angular momentum, which is usually denoted by the letter K, is conserved. This symmetry also implies that $\Im_{x'} = \Im_{y'}$. We will use the symbol $\Im$ to denote this moment of inertia.

In the intrinsic system the rigid body is at rest and here the symmetry is only cylindrical. But in the laboratory system the symmetry is spherical because the rigid body rotates in all directions and there is no way to distinguish one of these directions from another. Therefore the angular momentum $\mathbf{J}$ is conserved in the laboratory system and that implies, as we have seen in Chapter 1, that J^2 and J_z are good quantum numbers

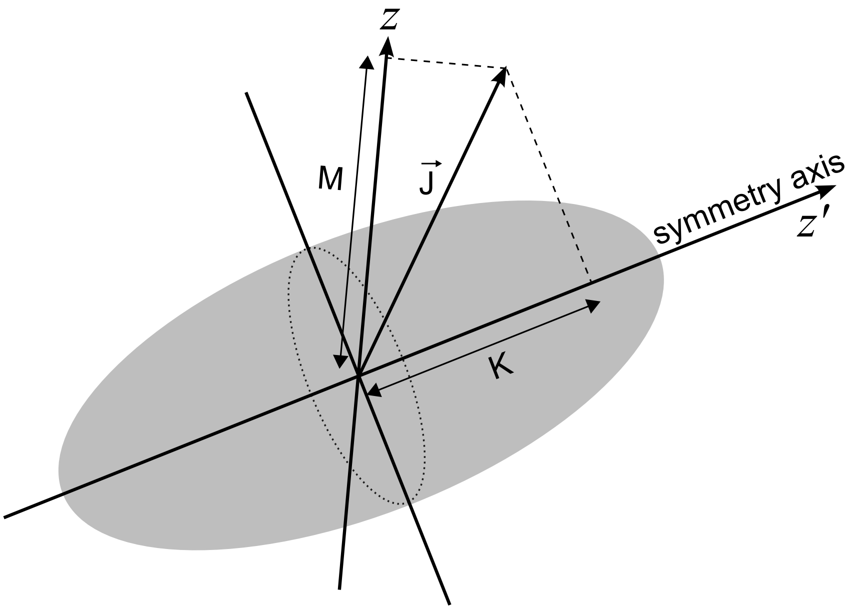

Figure 5.1: A cylindrically symmetric rotator. The symmetry axis is z'. The projection of the angular momentum upon this axis is K, and upon the z-axis in the laboratory frame is M.

which can be used to label the rotational state. The quantum number associated with J_z is usually called M. The state thus carries J, M, K as quantum numbers, i. e. it can be written as $|JMK\rangle$ or,

$$D^J_{MK}(\theta, \phi, \varphi) = \langle \theta\phi\varphi | JMK \rangle \tag{5.2}$$

which are called "d-functions". They satisfy the eigenvalue equations,

$$\begin{aligned} \boldsymbol{J}^2 D^J_{MK}(\theta, \phi, \varphi) &= \hbar^2 J(J+1) D^J_{MK}(\theta, \phi, \varphi), \\ J_z D^J_{MK}(\theta, \phi, \varphi) &= \hbar M D^J_{MK}(\theta, \phi, \varphi), \\ J_{z'} D^J_{MK}(\theta, \phi, \varphi) &= \hbar K D^J_{MK}(\theta, \phi, \varphi) \end{aligned} \tag{5.3}$$

In classical mechanics one can distinguish whether an spherically symmetric body rotates or not, but in quantum mechanics all directions are the same and the body appears to be at rest. Therefore there is no rotational energy associated to such a system or, in general, to degrees of freedom corresponding to a cylindrical symmetry. The Hamiltonian (5.1) thus becomes,

$$H = \frac{J^2_{x'} + J^2_{y'}}{2\Im} = \frac{J^2 - J^2_{z'}}{2\Im} \tag{5.4}$$

The eigenfunctions of this Hamiltonian satisfying all the symmetries of the system, including the cylindrical symmetry, the reflection symmetry with respect to the (x',y') plane and the parity symmetry (notice that the Hamiltonian is invariant under the parity operation of changing $\mathbf{r}$ by $-\mathbf{r}$) is,

$$\langle \theta\phi\varphi | JMK \rangle = c \left(D^J_{MK} + (-1)^J D^J_{M-K} \right) \tag{5.5}$$

where c is a normalization constant. The corresponding eigenenergies are, from Eq. (5.4),

$$E(J, K) = \hbar^2 \frac{J(J+1) - K^2}{2\Im} \tag{5.6}$$

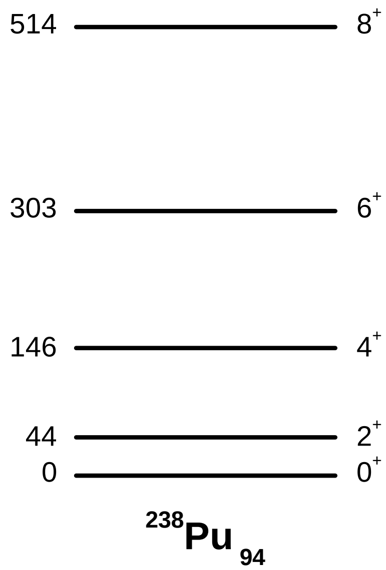

Figure 5.2: Ground rotational band (i. e. K=0) of ^{238}Pu. Energies are in keV. The rotational energies $E(J,0) = E(2,0)\ J(J+1)/6$, where $E(2,0) = 44keV$ in this case, are $E(4,0)$=147 keV, $E(6,0)$=308 keV and $E(8,0)$=528 keV.

Since $J \geq K$ and positive for a given $K \geq 0$ there is a band of states with energies proportional to $J(J+1)$. This is call "rotational band". The lowest lying of these bands is the one corresponding to $K = 0$, and its lowest state, i. e. $E(0,0) = 0$, is the ground state of the rotational nucleus. Therefore this band is called ground state band. From Eq. (5.5) one sees that for the ground state band $\langle\theta\phi\varphi|JMK\rangle$ vanishes if J is odd. Therefore the members of the ground state band have even J and parity $(-1)^J = +1$ (since parity is conserved) and the energy is

$$E(J,0) = \hbar^2 \frac{J(J+1)}{2\mathfrak{I}} \tag{5.7}$$

This is a very characteristic spectrum and many nuclei follows it. An example is the nucleus ^{238}Pu which spectrum is shown in Fig. 5.2.

An important sign that a spectrum corresponds to a ground state band is that the relation between the energies of the first 4^+ and 2^+ states should be 10/3, as shown by Eq. (5.7). In general, the levels of a rotational ground band are related to the energy of the first excited state 2^+ by the relation $E(J,0) = E(2,0)\ J(J+1)/6$. The agreement between theory and experiment can be excellent, as is seen in Fig. 5.2. The moment of inertia of a rotational nucleus can be extracted from the experimental energy $E(2,0)$, since

$$\mathfrak{I} = \hbar^2 \frac{J(J+1)}{2E(2,0)}. \tag{5.8}$$

For the $K > 0$ bands one can show that K must be even and J=K, K+1, K+2,... The lowest of the states of a band is called bandhead, and it corresponds to J=K. Therefore the bandhead energy E_{bh} is given by,

$$E_{bh}(K) = \hbar^2 \frac{K}{2\mathfrak{I}} \tag{5.9}$$

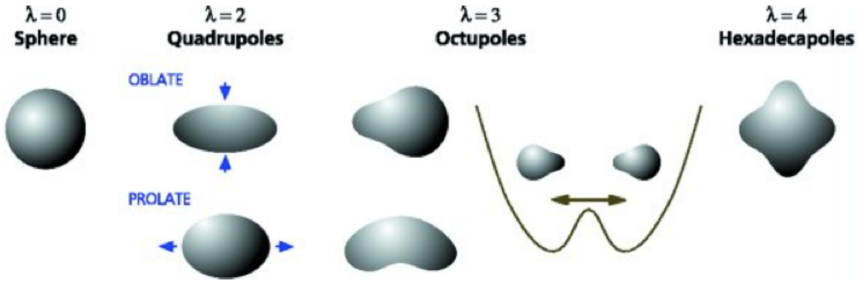

Figure 5.3: The different nuclear shapes that can be parametrised by spherical harmonic functions, where λ characterises the different orders of the corresponding distributions.

The rotational model has been very successful to explain the spectra of a large number of nuclei. It was formulated by A. Bohr and B. Mottelson in the beginning of the 1950's and it was corroborated experimentally by J. Rainwater. They were awarded the 1975 Nobel Prize in Physics for this work.

This model complements the shell model since it works very well in the middle of the major shells of Fig. 3.2, i. e. with values of the number of neutrons and protons which are far from magic numbers. However the shell model is, more than a model, a procedure to obtain a very good representation to solve the nuclear many-body problem. Therefore there have been many attempts to explain rotational spectra by using the shell model. This, which is even been pursued at present, has had a great importance in the understanding of nuclear correlations.

5.2 Deformed shell model (Nilsson model)

The spherical shell model can explain many features of spherical nu- clei, but needs modifying to describe nuclei with many nucleons outside a closed shell. The residual interaction between these many valence nucleons may be more simply described by a deformed potential. For nuclear rotation to be observable, the nuclei have to be non-spherical, so that they have a preferred axis.

Quadrupole ($\lambda = 2$) deformations can give rise to asymmetric shapes. These triaxial distortions are governed by the γ shape degree of freedom, and this describes a stretching/squashing effect at right angles to the major nuclear axis. Gamma is measured in degrees, where $\gamma = 0°$ and $\gamma = 60°$ correspond to prolate and oblate shapes respectively. Completely triaxial shapes have $\gamma = 30°$.

The model that describes axially symmetric nuclei is called the Deformed Shell Model. In this model the Schrödinger equation is solved using the potential that describes the shape of the nucleus. An result of the deformation is that the orbital angular momentum, l, and the intrinsic spin, s, are no longer good quantum numbers and thus, states with different l-values but the same parity can mix. The energy of the states now depends on the component of the single-particle angular momentum (j) along the symmetry axis, which is denoted by Ω. For each orbital with angular momentum j, there are 2j+1 values of Ω ($= m_j$ in the absence of other couplings). However, levels with $+\Omega$ and Ω have the same energy due to the reflection symmetry of axially symmetric nuclei, so that each state is now doubly degenerate, i.e. two particles can be placed in each state. For example the f7/2 orbital can have $|\Omega|$ equal to 7/2, 5/2, 3/2 and 1/2. The ordering of these Ω levels depends on the particular shape of the nucleus since the lowest in energy is the orbital which interacts (overlaps) the most with the nuclear core. For prolate shaped nuclei the states with the lowest Ω values are the most tightly bound, whereas for oblate shaped nuclei, the states with the highest Ω occur lowest in energy. Such deformed shell model calculations were first performed in 1955 by Nilsson with an anisotropic harmonic oscillator potential and the calculated states (called Nilsson orbitals) are labelled by $\Omega[Nn_z\Lambda]$ where N is the total oscillator shell quantum number and determines the parity, given by $(-1)^N$. Lambda is the projection of the particle orbital angular momentum, l, on the nuclear symmetry axis, and n_z is the number of oscillator shell quanta along the direction of the symmetry axis.

Homeworkproblems 5

Exercise 1:

Determine the moment of inertia from the experimental spectrum corresponding to the rotational ground state band in

(a) ^{238}Pu.

(b) ^{168}Yb.

Hint: Experimental data could be found in http://www.nndc.bnl.gov/nudat2/