EL2520
Control Theory and Practice

Lecture 10:
Glover-McFarlane loop shaping and controller order reduction

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Today's lecture

- Glover McFarlane loop shaping
  - Robustifying controller "around" nominal design
  - A design example
- Simplification of control laws
  - Balanced truncation

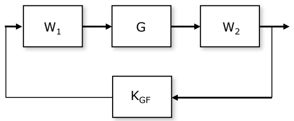
Course book chapters 10.5 and 3.6

Robustification of control laws

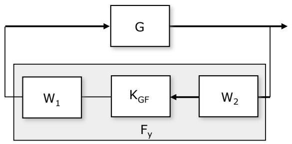
Three-step design:

1. Perform initial (e.g. lead-lag) design focusing on performance

2. A second step augments controller to create a robust design



3. Actual controller combines initial design and robustifying controller.



A robust stabilization problem

Write shaped plant $G_s(s) = W_2(s)G(s)W_1(s)$ as

$$G_s(s) = M(s)^{-1}N(s)$$

Find a controller that stabilizes

$$G_s(s) = (M(s) + \Delta_M(s))^{-1}(N(s) + \Delta_N(s))$$

for all uncertainties satisfying

$$\|\Delta_M(s)\ \Delta_N(s)\|_\infty \leq \epsilon$$

General perturbation

- could imply additional unstable poles and NMP zeros
- Tend to make limited changes to loop gain around cross-over

Co-prime factorization

Fact: Any transfer matrix can be co-prime factorized

$$G(s) = M(s)^{-1}N(s)$$

where the transfer matrices M and N are stable and co-prime.

The coprime factorization is not unique.

A coprime factorization is *normalized* if N, M satisfy

$$M(s)M(-s)^T + N(s)N(-s)^T = I$$

Normalized coprime factorizations are unique
(up to a multiplication with a unitary matrix)

Co-prime factorization cont'd

Example: The system

$$G(s) = \frac{(s-1)(s+2)}{(s-3)(s+4)}$$

has a coprime factorization given by

$$M(s) = \frac{s-3}{s+2}, \quad N(s) = \frac{s-1}{s+4}$$

but it is not normalized. Another factorization is

$$M(s) = \frac{(s-3)(s+4)}{s^2+k_1 s+k_2}, \quad N(s) = \frac{(s-1)(s+2)}{s^2+k_1 s+k_2}$$

This one is normalized for the appropriate values of k_1, k_2

Robust stabilization: solution

Consider a state-space representation of the shaped plant

$$\dot{x} = Ax + Bu, \quad y = Cx$$

1. Solve the Riccati equations

$$AZ + ZA^T - ZC^TCZ + BB^T = 0$$
$$A^TX + XA - XBB^TX + C^TC = 0$$

2. Let λ_m be the maximum eigenvalue of XZ, $\alpha \geq 1$ and introduce

$$\gamma = \alpha(1+\lambda_m)^{1/2}, \quad R = I - \frac{1}{\gamma^2}(I + ZX)$$
$$L = B^TX, \quad K = R^{-1}ZC^T$$

3. Then, the following controller stabilizes all plants with $\varepsilon \leq \gamma^{-1}$

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu + K(y - C\hat{x}), \quad u = -L\hat{x}$$

A link to LQG

Note that the LQG-optimal controller for the criterion

$$\int_{t=0}^{\infty} y(t)^T y(t) + u(t)^T u(t)\, dt$$

with v_1 acting on the input and $R_1 = I,\ R_2 = I,\ R_{12} = 0$ is

$$u(t) = -L\hat{x}(t), \quad \frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

where

$$L = B^TS \qquad A^TS + SA - SBB^TS + C^TC = 0$$
$$K = R^{-1}PC^T \qquad AP + PA^T - PC^TCP + B^TB = 0$$

Same as Glover-McFarlane, apart from the R-matrix.

- when $\alpha \to \infty$, $R \to I$ and the two coincide.

Example

The DC motor

$$G(s) = \frac{20}{s(s+1)}$$

has state-space representation

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 20 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

The Riccati equations have solutions

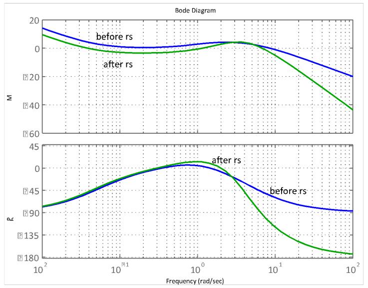
$$X = \begin{bmatrix} 0.32 & 0.05 \\ 0.05 & 0.15 \end{bmatrix}, \quad Z = \begin{bmatrix} 5.4 & 14.6 \\ 14.6 & 93.5 \end{bmatrix}$$

And XZ has eigenvalues 4.32 and 0.12.
Letting $\alpha = 1$, we find $\gamma = 2.54$, $L = \begin{bmatrix} 1 & 0.27 \end{bmatrix}$, $K = \begin{bmatrix} 27 & 102 \end{bmatrix}$

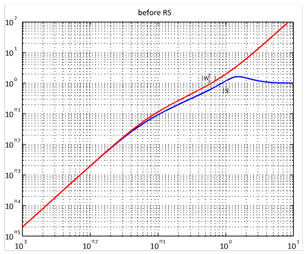
Example

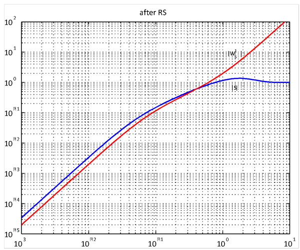
Robustifying lead-lag controller from Lecture 4

$$\gamma = 2.0\ ; \quad \tilde{F}_y = \frac{6.702s^4 + 50.47s^3 + 95.15s^2 + 51.52s + 2.134}{s^5 + 12.93s^4 + 62.16s^3 + 143.2s^2 + 80.13s + 3.655}$$

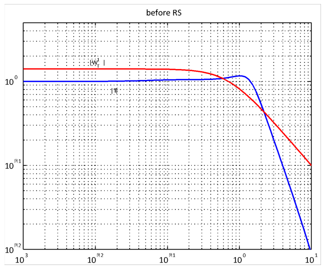


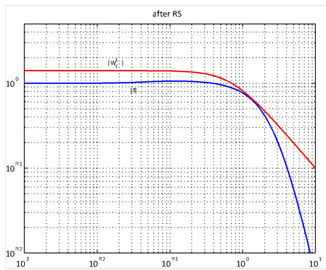
Example: effect on S





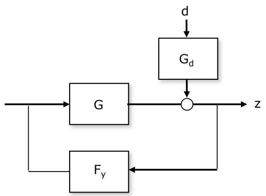
Example: effect on T





Example: a quick lead-lag design

Aim: controller with good disturbance rejection, cross-over around 10 rad/s.

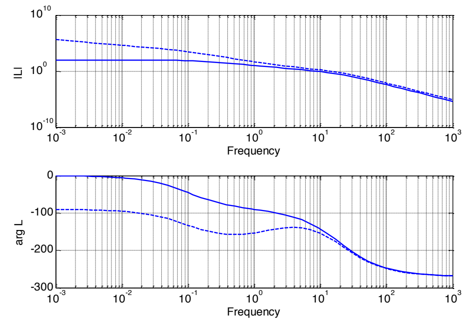


$$G = \frac{200}{10s+1}\frac{1}{(0.05s+1)^2}$$
$$G_d = \frac{100}{10s+1}$$

Since $z=(1+GF_y)^{-1}G_d$, we should aim for $F_y \approx G^{-1}G_d$.

Crude design: $F_y(s) = G^{-1}(0)G_d(0)\frac{s+a}{s}$, tune a to get desired cross-over

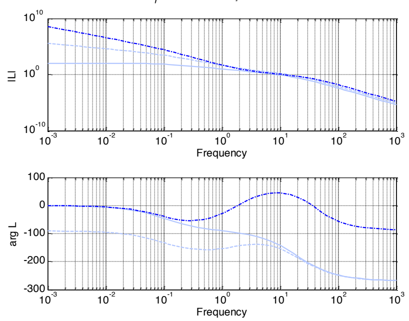
Loop gains for rough design



Low frequency gain, cross over ok, but poor phase margins.

Glover-McFarlane

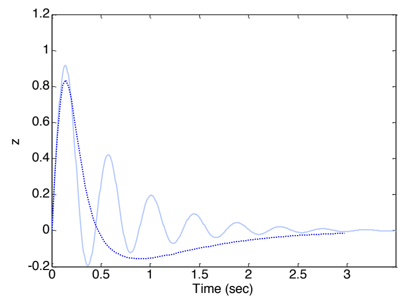
Rather than tweaking the lead-lag, we simply apply Glover-McFarlane
- optimization returns $\gamma = 2.33$, and total controller order 5 (why?)



Much improved stability margins!

Disturbance responses

Response in z to step in d for nominal and robustified lead-lag



Glover-McFarlane – MIMO recipe

1. Rearrange control inputs so that G close to diagonal (via RGA)
2. Choose W_2 to be a constant scaling matrix
3. Choose $W_1(s)$ to be a diagonal matrix and adjust the elements so that the singular values get desired shape

$$\omega < \omega_{BS}: \quad \bar{\sigma}(S(i\omega)) < W_S^{-1}(i\omega) \quad \Rightarrow \underline{\sigma}(L) > W_S(i\omega)$$
$$\omega > \omega_{BT}: \quad \bar{\sigma}(T(i\omega)) < W_T^{-1}(i\omega) \quad \Rightarrow \bar{\sigma}(L) < W_T^{-1}(i\omega)$$

   (cf. loop shaping lecture)
   Use decoupling only if necessary
4. Perform robust stabilization. If $\gamma_m > 4$, go back and modify $W_1(s)$

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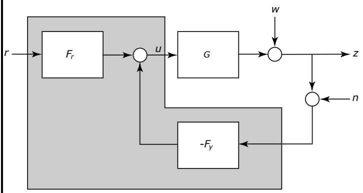
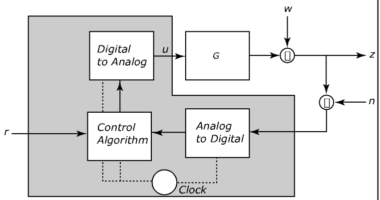
Digital implementation

Controller "on paper"… …and as real implementation





Two key steps:
- Simplification: reduce number of controller states
- Discretization: continuous-time → discrete-time control law (see book!)

Controller simplification

LQG, H_∞ and Glover-McFarlane designs give high-order controllers

Often interesting to reduce order (number of states) of controller
- Easier implementation
- Smaller computational delay
- …

but need to ensure that simplified controller is "similar" to original design

State-space realizations

Linear system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Can be represented in many ways (observable canonical form, controllable canonical form, ...) via change of variables

$$\zeta = Tx$$

Gives

$$\dot{\zeta}(t) = T\dot{x} = TAx + TBu = TAT^{-1}\zeta + TBu$$
$$y = CT^{-1}\zeta + Du$$

Balanced realizations: allow to quantify the relative importance of each state in describing the input-output behavior of the system

The Controllability Gramian

Measures how states are influenced by impulse input

$$u(t) = \delta(t),\ x(0) = 0 \Rightarrow x(t) = e^{At}B$$
$$S_x = \int_0^\infty x(t)x^T\,dt = \int_0^\infty e^{At}BB^Te^{A^Tt}\,dt$$

Note: matrix exponential

$$e^M = I + M + \frac{1}{2}M^2 + \frac{1}{3}M^3 + \dots$$

Gramian under change-of-coordinates

Express the Gramian in new variables:

$$u(t) = \delta(t),\ \zeta(0) = Tx(0) = 0 \Rightarrow \zeta(t) = e^{TAT^{-1}t}B$$

Exploiting the properties of the matrix exponential,

$$e^{TAT^{-1}t} = I + tTAT^{-1} + \frac{t^2}{2}TAT^{-1}TAT^{-1} + \cdots \qquad = Te^{At}T^{-1}$$

So

$$S_\zeta = \int_0^\infty \zeta(t)\zeta^T(t)\,dt = \int_0^\infty e^{TAT^{-1}t}TBB^TT^Te^{TAT^{-1}t}\,dt =$$
$$= TS_xT$$

Fact: Can pick T so that the Gramian is diagonal

The Observability Gramian

Measures how different states contribute to output energy

$$u(t) = 0,\ x(0) = x_0 \Rightarrow y(t) = Ce^{At}x_0$$

The *observability gramian*

$$\int_0^\infty y(t)^Ty(t)\,dt = x_0^T\left[\int_0^\infty e^{A^Tt}C^TCe^{At}\,dt\right]x_0 = x_0^TO_xx_0$$

Change of coordinates gives

$$O_\zeta = T^{-T}O_xT^{-1}$$

Fact: can pick T so that *both* observability and controllability gramians for new variables are equal and diagonal.

$$O_\zeta = S_\zeta = \Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_n)$$

Balanced truncation

Idea: states with small Hankel singular values (σ_i) have small influence on input-output behaviour, could be eliminated

Partition transformed state-vector into one part with large σ_i's (to keep) and one part with small σ_i's (to eliminate).

$$\begin{bmatrix} \dot{\zeta}_1 \\ \dot{\zeta}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} + Du$$

Balanced truncation: remove all parts associated with ζ_2.
Balanced residualization: set $\dot{\zeta}_2(t) = 0$, and eliminate ζ_2-components.

Balanced truncation error bounds

Fact: Let G^r be the reduced system, obtained by keeping m states and eliminating the n-m remaining. Then

$$\sigma_{m+1} \leq \|G - G^r\|_\infty \leq 2 \sum_{i=m+1}^{n} \sigma_i$$

Learn more in our graduate course ”Introduction to model reduction”

Balanced truncation

1. Compute balanced realization, including T and Gramian Σ
2. Plot Hankel singular values (diagonal elements of Σ). Eliminate states with very small values (compare error bound)

Example. Robustified lead-lag for DC-motor is of order eight. Its Hankel singular values are

$$\begin{bmatrix} \infty & 1.0122 & 0.5296 & 0.1479 & 0.0029 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

so a fifth order controller seems (and is) appropriate!

Summary

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- A design example
- Simplification of control laws
  - Balanced truncation