The BCS Model

Sara Changizi

This presentation closely follows parts of chapter 6 in Ring & Schuck “The nuclear many-body problem”.

Outline

- Introduction to pairing
  - Essential experimental facts
- The BCS model
  - Pure pairing force

Introduction to pairing

- In solid-state physics: A pair of electrons in metal with opposite spins and momentums close to Fermi surface interact with each other and make a pair. They have energy lower than Fermi surface which indicates that they are bound.
- BCS: Bardeen-Cooper-Schrieffer: microscopic model describes superconductivity (Nobel prize 1972)
- In nuclear physics: A pair of nucleons with total spin $I_Z=0$ $(m_i,-m_i)$. This is a short range (large spatial overlap) nucleon-nucleons interaction.

Experimental observations

- Experimental observations that require short range interaction in the model :
  - The energy gap
  - The level density
    - Higher level density in low-lying excitation energies than found experimental values
  - Odd-even mass effect $$M_{(A\ odd)} > \frac{M_{A-1} + M_{A+1}}{2}$$
  - Moment of inertia
    - Lowering of the moment of inertia compared to rigid body value
  - The low-lying 2^+ in even nuclei
    - Vibrate with low frequency :quadrupole oscillations

The energy gap due to pairing

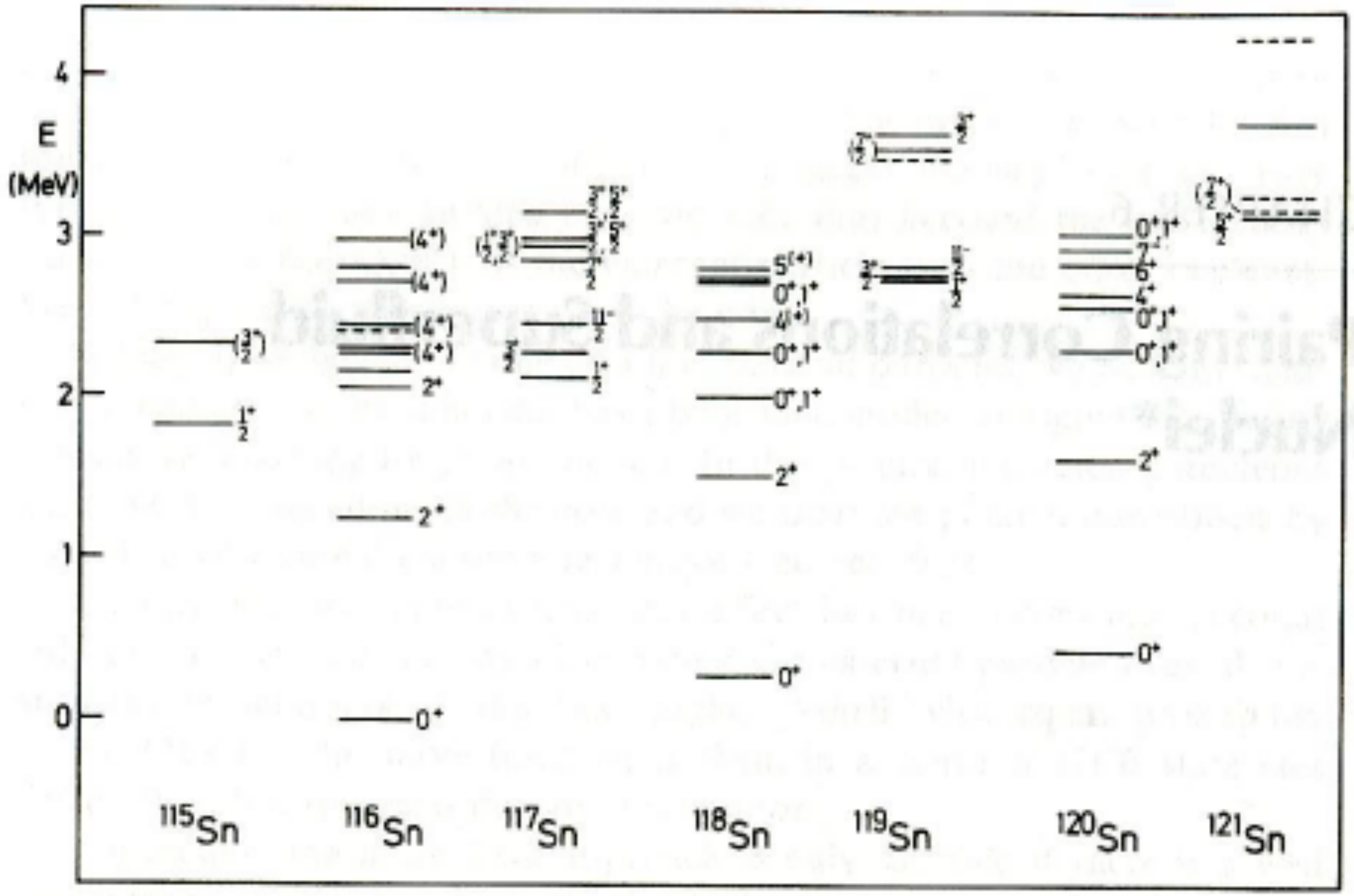


Figure 6.1. Excitation spectra of the $_{50}$Sn isotopes.

Introduction to pairing

- To explain these phenomena we need to take into the account short range nucleon-nucleon interaction.
- The most effective pairing coupling is $I=0$
- No exact solution, it is an approximation by variational principle

Many-body system

A many-body system is described by following Hamiltonian in second-quantization form (particle-number representation):

$$\boxed{\begin{aligned} &H = \sum_{k_1 k_2} t_{k_1 k_2} a^{+}_{k_1} a_{k_2} + \frac{1}{2} \sum_{k_1 k_2 k_3 k_4 <>0} v_{k_1 k_2 k_3 k_4} a^{+}_{k_1} a^{+}_{k_2} a_{k_4} a_{k_3} \\ &t_{k_1 k_2} \equiv \langle 1:k_1 | t_1 | 1:k_2 \rangle; \quad v_{k_1 k_2 k_3 k_4} \equiv \langle 1:k_1; 2:k_2 | v(1,2) | 1:k_3; 2:k_4 \rangle \end{aligned}}$$

First term: one body operator in second-quantization
Second term: two body operator in second-quantization
V=matrix elements of the nucleon-nucleon interaction
K_i are single-particle states and run over all available stats
Sums do not run on particles but on the infinite set of one-body states

BCS model- BCS state

In BCS model we speculate that ground state should be built up from pair creation operators $a^+_k a^+_{-k}$

This is the approximated solution:
Trial wave function for even-even nuclei

$$|BCS\rangle = \prod_{k>0}\left(u_k + v_k a_k^+ a_{\bar{k}}^+\right)|-\rangle$$

$$|HF\rangle = \prod_{\alpha=1}^{N} a_\alpha^+ |-\rangle$$

K are the single particle levels
V_k^2 and U_k^2represent the probability that a certain pair state $\{k,-k\}$
Is or is not occupied.

$$|k\rangle = |nljm\rangle,$$

An example is a spherical basis

$$Conjugate\ state: |\bar{k}\rangle = |nlj-m\rangle\ m>0$$

BCS model- BCS state

Example: For Hartree-Fock states we would have:
v_k=1 and u_k=0 below Fermi level
v_k=0 and u_k=1 above Fermi level
However in BCS model states over Fermi level can be occupied (energetically favored)

u_k and v_k are variational parameters. We determine them in a way that the corresponding state has minimum energy.
Normalization of BCS state:

$$\langle BCS | BCS \rangle = \langle 0 | \prod u_k + v_k a_{\bar{k}} a_k \prod u_{k'} + v_{k'} a_{k'}^{+} a_{\bar{k}'}^{+} | 0 \rangle = \prod_{k>0} \left(u_k^2 + v_k^2 \right)$$

We require: $\boxed{u_k^2 + v_k^2 = 1}$

BCS model

Great disadvantage: Particle number is not conserved in BCS! BCS state is the superposition of different number of pairs.

$$\left|BCS\right\rangle \propto \left|-\right\rangle + \sum_{k>0}\frac{v_k}{u_k}a_k^+a_{-k}^+\left|-\right\rangle + \frac{1}{2}\sum_{kk'}\frac{v_kv_{k'}}{u_ku_{k'}}a_k^+a_{-k}^+a_{k'}^+a_{-k'}^+\left|-\right\rangle + \ldots$$

$$N = \left\langle BCS\middle|\hat{N}\middle|BCS\right\rangle = \left\langle BCS\middle|a_k^+a_k + a_{\bar{k}}^+a_{\bar{k}}\middle|BCS\right\rangle = 2\sum_{k>0}v_k^2$$

This is fit to the interpretation of v_k

Particle number uncertainty $\quad (\Delta N)^2 = \left\langle BCS\middle|\hat{N}^2\middle|BCS\right\rangle - N^2 = 4\sum_{k>0}u_k{}^2v_k{}^2$

BCS model

Hence, we need to restrict the variation by a supplementary condition. We define a parameter λ in the Hamiltonian to keep the expectation value of particle number to the desired particle number.

We add a term –λN to Hamiltonian: $$\hat{H}' = \hat{H} - \lambda\hat{N}$$

We call this Lagrange multiplier λ as Fermi energy or chemical energy, since it describes the energy variation in the system by changing the particle number.

BCS model-Pure pairing force

Hamiltonian has a the form:

$$\sum_{k>0} \varepsilon_k a_k^+ a_k + \sum_{kk'>0} \langle k,\bar{k} | v | k',\bar{k}' \rangle a_k^+ a_{\bar{k}}^+ a_{\bar{k}'} a_{k'}$$

Single-particle part $\quad v_{k\bar{k}k'\bar{k}'}$ Residual interaction acting only on pairs of nucleons

In this model we assume a constant matrix elements – G (pure pairing force):

$$\boxed{H = \sum_{k>0} \varepsilon_k a_k^+ a_k - G \sum_{kk'>0} a_k^+ a_{\bar{k}}^+ a_{\bar{k}'} a_{k'}}$$

BCS model- Pure pairing force

Lets consider the Hamiltonian with the variational condition

$$H = \sum_{k>0} \varepsilon_k a_k^+ a_k - G \sum_{kk'>0} a_k^+ a_{\bar{k}}^+ a_{\bar{k}'} a_{k'}$$

$$H' = H - \lambda \hat{N}$$

$$\langle BCS | H - \lambda \hat{N} | BCS \rangle = \langle BCS | \sum_k \left(\varepsilon_k^0 - \lambda \right) a_k^+ a_k - G \sum_{kk'} a_k^+ a_{-k}^+ a_{-k'} a_{k'} | BCS \rangle$$

The expectation value:

$$\boxed{\Delta = G \sum_{k>0} u_k v_k}$$

$$\langle BCS | \hat{H}' | BCS \rangle =$$

$$2 \sum_{k>0} (\varepsilon_k^0 - \lambda) \underbrace{v_k^2}_{\langle BCS | a_k^+ a_k | BCS \rangle} - \sum_{k>0} G \underbrace{v_k^4}_{\langle BCS | a_k^+ a_{-k}^+ a_{-k} a_k | BCS \rangle} \underbrace{- G \left(\sum_{k>0} \underbrace{u_k v_k}_{\langle BCS | a_k^+ a_{-k}^+ a_{-k'} a_{k'} | BCS \rangle} \right)^2}_{\Delta}$$

BCS model- Pure pairing force

v_k determines the BCS wave function completely, we can express u_k in terms of v_k by the normalization condition

$$\delta\langle BCS|\hat{H}'|BCS\rangle = 0,$$

$$\left.\begin{array}{l} \left(\dfrac{\partial}{\partial v_k} + \dfrac{\partial u_k}{\partial v_k}\dfrac{\partial}{\partial u_k}\right)\langle BCS|H'|BCS\rangle = 0 \\ u_k \partial u_k + v_k \partial v_k = 0 \Rightarrow \dfrac{\partial u_k}{\partial v_k} = -\dfrac{v_k}{u_k} \end{array}\right\} \left(\frac{\partial}{\partial v_k} - \frac{v_k}{u_k}\frac{\partial}{\partial u_k}\right)\langle BCS|H'|BCS\rangle = 0$$

$$\left(\frac{\partial}{\partial v_k} - \frac{v_k}{u_k}\frac{\partial}{\partial u_k}\right)\left(2\sum_{k>0}(\varepsilon_k^0 - \lambda)v_k^2 - \sum_{k>0} G v_k^4 - G\left(\sum_{k>0} u_k v_k\right)^2\right) = 0$$

BCS model

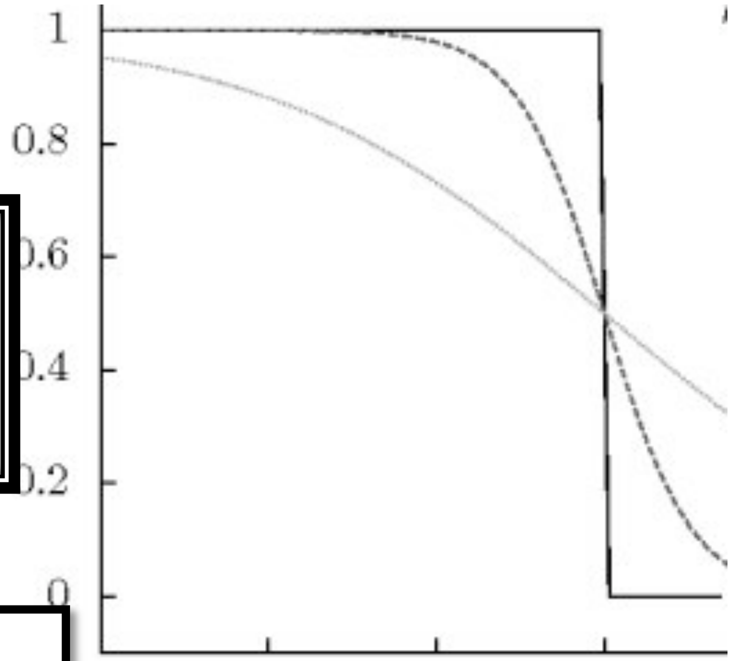
We get for BCS equations the following:

$$\varepsilon_k = \varepsilon_k^0 - \lambda - \underbrace{G \cdot v_k^2}_{often\ neglected}$$

$$\boxed{2\varepsilon_k v_k u_k + \Delta(v_k^2 - u_k^2) = 0}$$

The occupation probability in non interacting and
In the interacting case (assume we know Δ):

$$\boxed{v_k^2 = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{\Delta^2}{\varepsilon_k^2 + \Delta^2}}\right) = \frac{1}{2}\left(1 \pm \frac{\varepsilon_k}{\sqrt{\varepsilon_k^2 + \Delta^2}}\right)}$$



Insert this into definition of gap we can get this iterative equation.

Gap equation:
$$\boxed{\Delta = \frac{G}{2}\sum_{k>0}\frac{\Delta}{\sqrt{\varepsilon_k^2 + \Delta^2}}}$$

Results of BCS model

- In the special case of a single j-shell: all ε_k are equals so all v_k^2's are equals. From the particle-number condition we get

$$\left\langle BCS \middle| \hat{N} \middle| BCS \right\rangle = 2\sum_{k>0} v_k^{\,2} = N \Rightarrow v_k = \sqrt{\frac{N}{2\Omega}}, u_k = \sqrt{1-\frac{N}{2\Omega}}$$

$$\Delta = G \cdot \sqrt{\frac{N}{2}\left(\Omega - \frac{N}{2}\right)}$$

- The gap has a parabolic dependence on the number of particle in the shell. It is zero for empty or filled shells. For N=Ω: $2\Delta = G \cdot \Omega$

Ω is the number of pairs

Thanks!