2E1252
Control Theory and Practice

Lecture 9:
LQG design example and relation to H_2

Mikael Johansson
School of Electrical Engineering
KTH, Stockholm, Sweden

Today's lecture

- Linear quadratic control review
- A design example: radial control of DVD servo
- Relation to H_2-optimal control

Mikael Johansson mikaelj@ee.kth.se

Linear quadratic Gaussian control

Model: linear system with white noise

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) + Nv_1(t)$$
$$y(t) = Cx(t) + v_2(t)$$
$$z(t) = Mx(t)$$

where v_1, v_2 are white noise with

$$\text{cov}([v_1,\ v_2]) = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$$

Objective: minimize effect of v on z, punish control cost

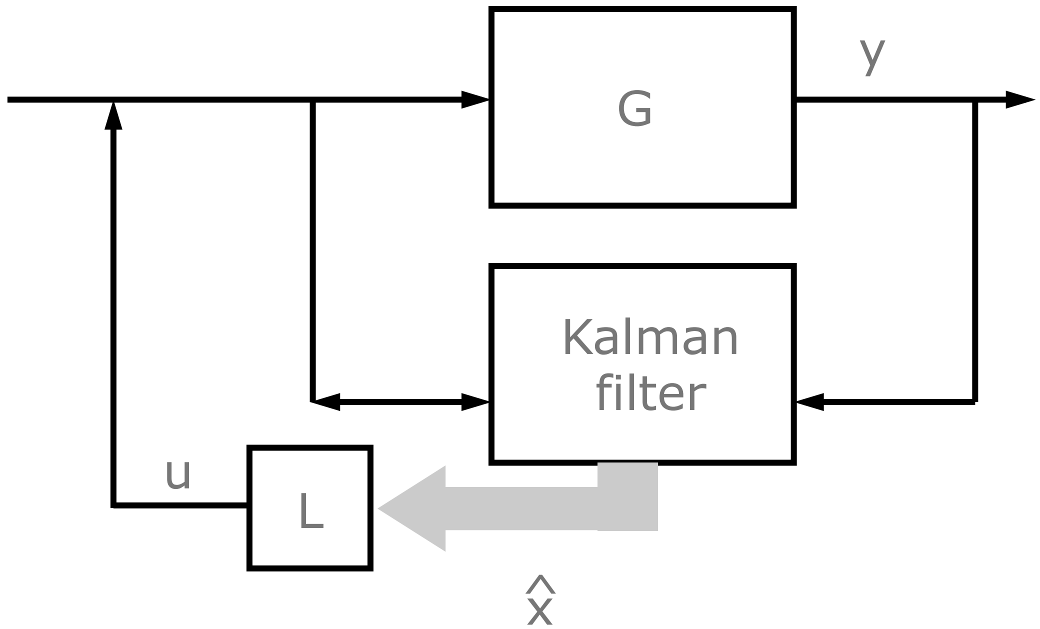
$$J = \mathbf{E}\left\{ \lim_{T\to\infty} \int_0^T [z^T Q_1 z + u^T Q_2 u]\ dt \right\}$$

LQG: Linear system, Quadratic cost, Gaussian noise

Solution structure

Optimal solution satisfies *separation principle*, composed of

- Optimal linear state feedback (Linear-quadratic regulator)
- Optimal observer (Kalman filter)



Mikael Johansson mikaelj@ee.kth.se

Optimal solution

State feedback

$$u(t) = -L\hat{x}(t) = -Q_2^{-1} B^T S \hat{x}(t)$$

where S is the solution to the algebraic Riccati equation

$$A^T S + SA + M^T Q_1 M - SBQ_2^{-1} B^T S = 0$$

Kalman filter

$$\hat{x}(t) = Ax(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

where $K=(PC^T+NR_{12})R_2^{-1}$ and P is the solution to

$$AP + PA^T + NR_1 N^T - (PC^T + NR_{12}) R_2^{-1} (PC^T + NR_{12})^T = 0$$

Example: LQR for scalar system

Scalar linear system

$$\dot{x}(t) = ax(t) + u(t), \qquad y(t) = x(t)$$

with cost

$$J = \int_0^\infty [x^2 + \rho u^2] \, dt$$

Riccati equation

$$2as + 1 - s^2/\rho = 0$$

has solutions

$$s = a\rho \pm \sqrt{(\rho a)^2 + \rho}$$

so the optimal feedback law is

$$u = -(s/\rho)x = -(a + \sqrt{a^2 + 1/\rho})x$$

Example: LQR for scalar system

Closed loop system

$$\dot{x}(t) = -(\sqrt{a^2 + 1/\rho})x(t)$$

Note: if system is unstable (a>0), then

- if control is expensive $\rho \to \infty$ then the minimum control input to stabilize the plant is obtained with the input u=-2|a|x, which moves the unstable pole to its mirror image –a
- if control is cheap ($\rho \to 0$), the closed loop bandwidth is roughly $1/\sqrt{\rho}$

Example: scalar system Kalman filter

Scalar linear system

$$\dot{x}(t) = ax(t) + u(t) + v_1(t), \qquad y(t) = x(t) + v_2(t)$$

with covariances $E\{v_1^2\}=R_1$, $E\{v_2^2\}=R_2$, $E\{v_1 v_2\}=0$.

Riccati equation

$$2ap + r_1 - p^2/r_2 = 0$$

gives

$$k = a + \sqrt{a^2 + r_1/r_2}$$

and estimation error dynamics

$$\frac{d}{dt}\tilde{x}(t) = -(\sqrt{a^2 + r_1/r_2})\tilde{x}$$

Interpretation: measurements discarded if too noisy.

The tuning knobs

State and control weights:

- Trade-off between control effort and state errors
  (Q_1=1, Q_2=ρ gives closed-loop bandwidth $\sim 1/\sqrt{\rho}$)
- Rule of thumb: start with diagonal Q_1, i.e.

$$z^T Q_1 z = q_{11} z_1^2 + \cdots + q_{kk} z_k^2$$

  where q_{ll} is inversely proportional to maximum allowed value of z_l
  (similarly with Q_2)

Noise covariance matrices

- Trade-off between sensitivity to process and measurement noise

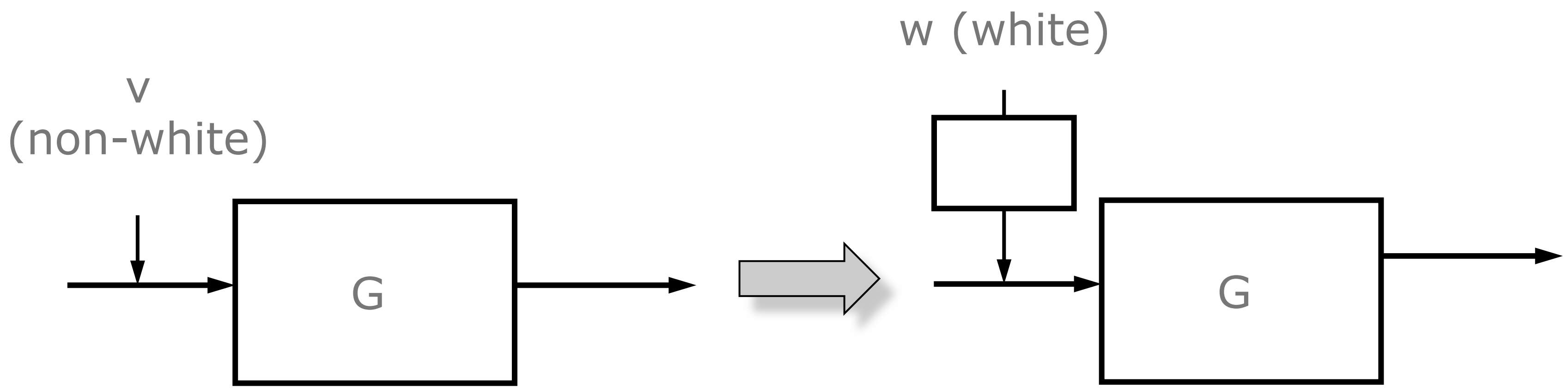
  (estimator bandwidth $\sim \sqrt{r_1/r_2}$)

Mikael Johansson mikaelj@ee.kth.se

White noise inputs

No serious restriction:

- practically all disturbance spectra can be realized as filtered white noise
- need to augment system model with disturbance model



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The servo problem

Preferred way: augment system with reference model

$$\frac{d}{dt}x_{\mathsf{ref}}(t) = A_{\mathsf{ref}}x_{\mathsf{ref}}(t) + Br_z(t)$$
$$e(t) = x_{\mathsf{ref}}(t) - x(t)$$

where the reference model states are measurable, and use

$$J = \mathbf{E}\left\{\lim_{T\to\infty}\int_0^T [e^TQ_1e + u^TQ_2u]\,dt\right\}$$

However, when r is assumed to be constant the solution is

$$u(t) = -L\widehat{x}(t) + L_{\mathsf{ref}}r_z(t)$$

where L_{ref} is determined so that static gain of closed-loop ($r_z \rightarrow z$) equals the identity matrix

Mikael Johansson mikaelj@ee.kth.se

LQG and loop shaping

- LQG: simple to trade-off response-time vs. control effort
  - but what about sensitivity and robustness?
- These aspects can be accounted for using the noise models
  - Sensitivity function: transfer matrix w→z
  - Complementary: transfer matrix n→z

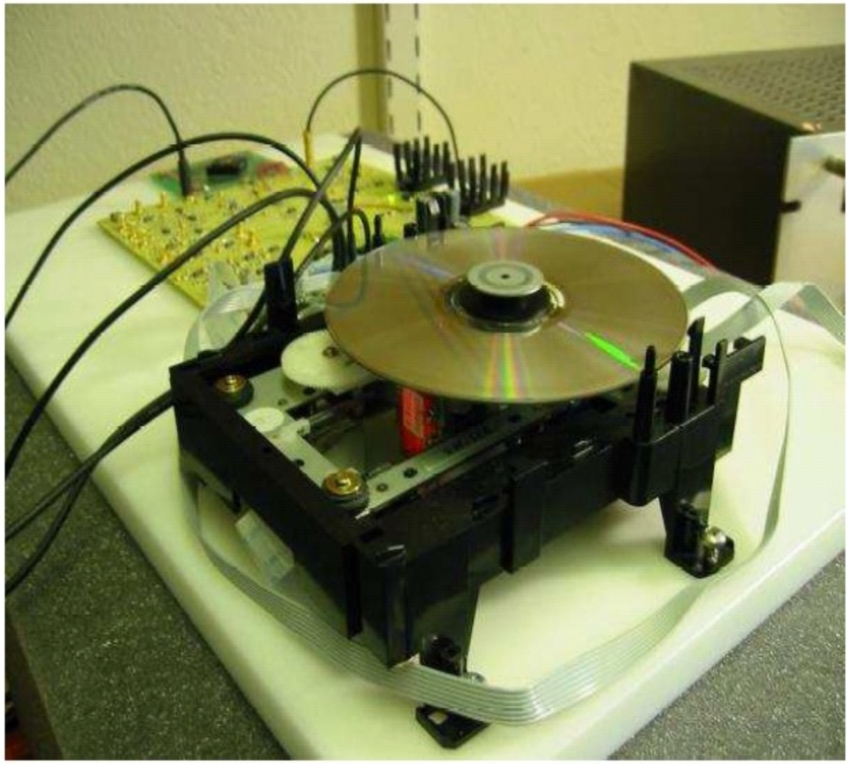
Example: S forced to be small at low frequencies by letting (some component of) w1 affect the input, and let w1 have large energy at low frequencies,

$$w_1(t) = \frac{1}{p+\delta} v_1(t)$$

(delta small, strictly positive, to ensure stabilizability)

Design example

Track following (radial control) in DVD player

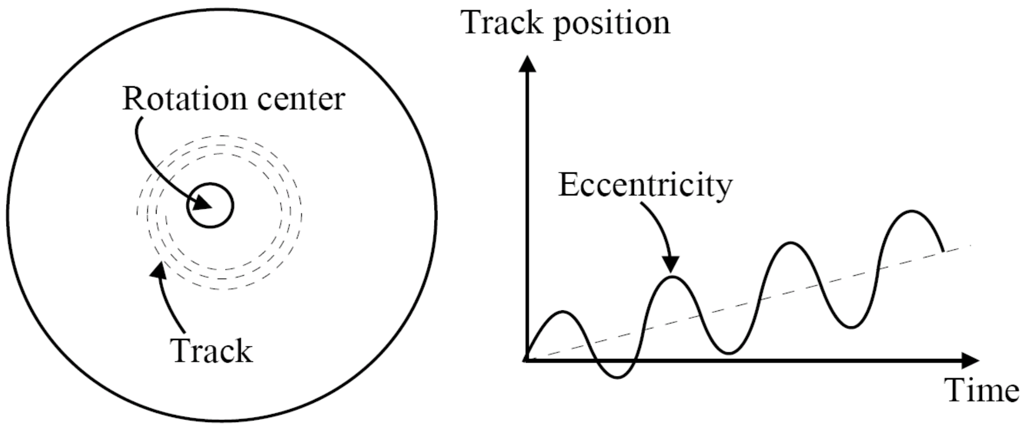


(based on laboratory exercise at LTH)

Design example

Control lens position to follow track

- High bandwidth to allow fast read/write
- Key challenge: eccentricity of tracks on disk
- Sinusoidal disturbance of order 100 track widths!

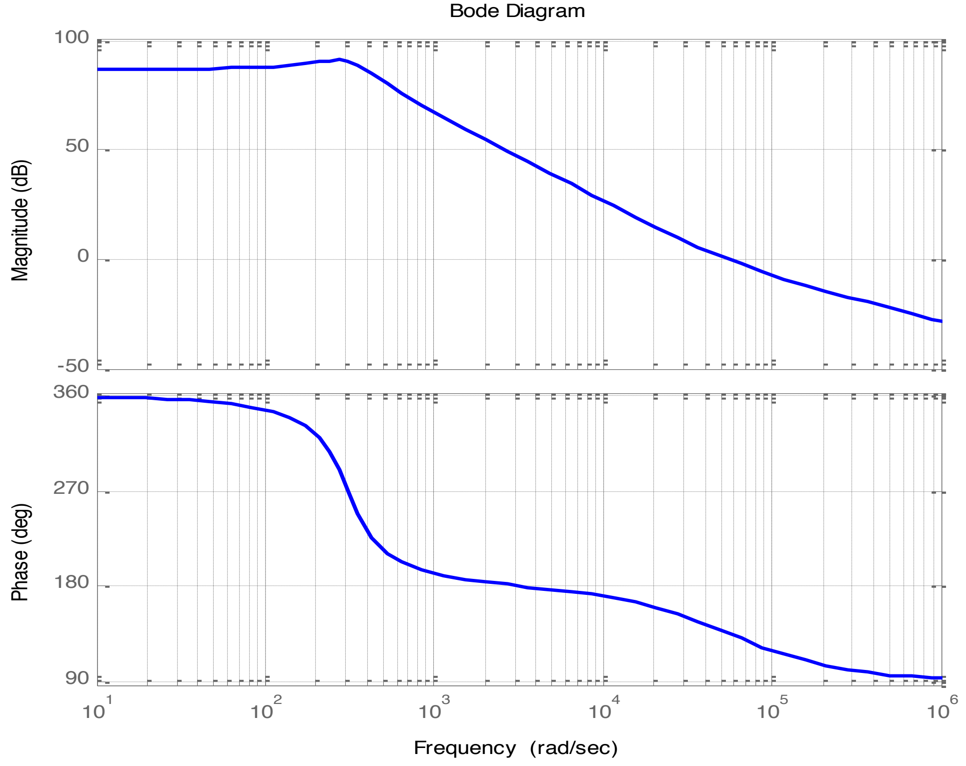


 Mikael Johansson mikaelj@ee.kth.se

Model

Model identified from real system

$$\dot{x} = \begin{pmatrix} 13.48 & -613.3 \\ 160.4 & -221.7 \end{pmatrix} x + \begin{pmatrix} -9.57 \\ -1046 \end{pmatrix} u$$
$$y = \begin{pmatrix} 3354 & 5.40 \end{pmatrix} x$$

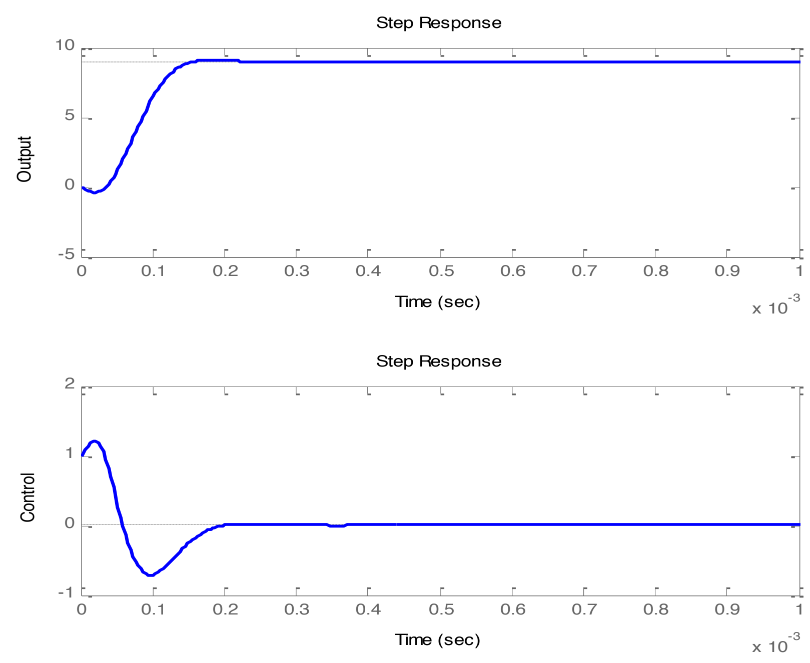


An initial design

Use

$$z = y, \quad Q_1 = 1, \quad Q_2 = 1, \quad R_1 = I, \quad R_2 = 1$$

consider response to unit step input in reference:

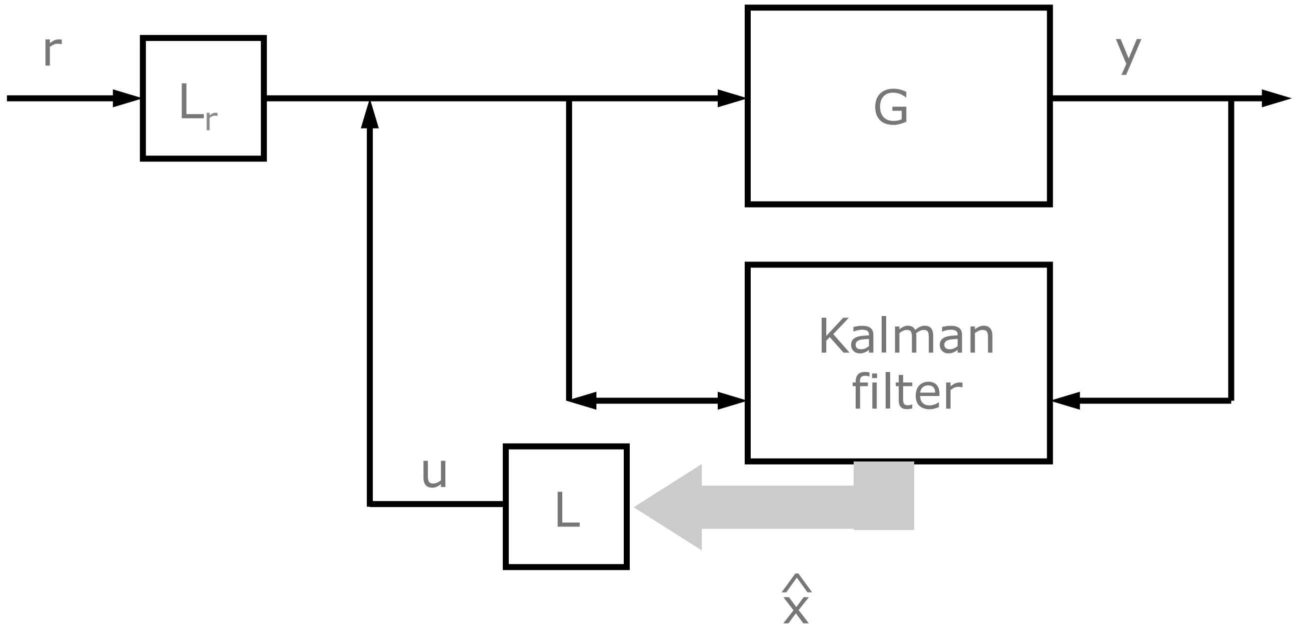


What is wrong?

Adjusting feedforward gain

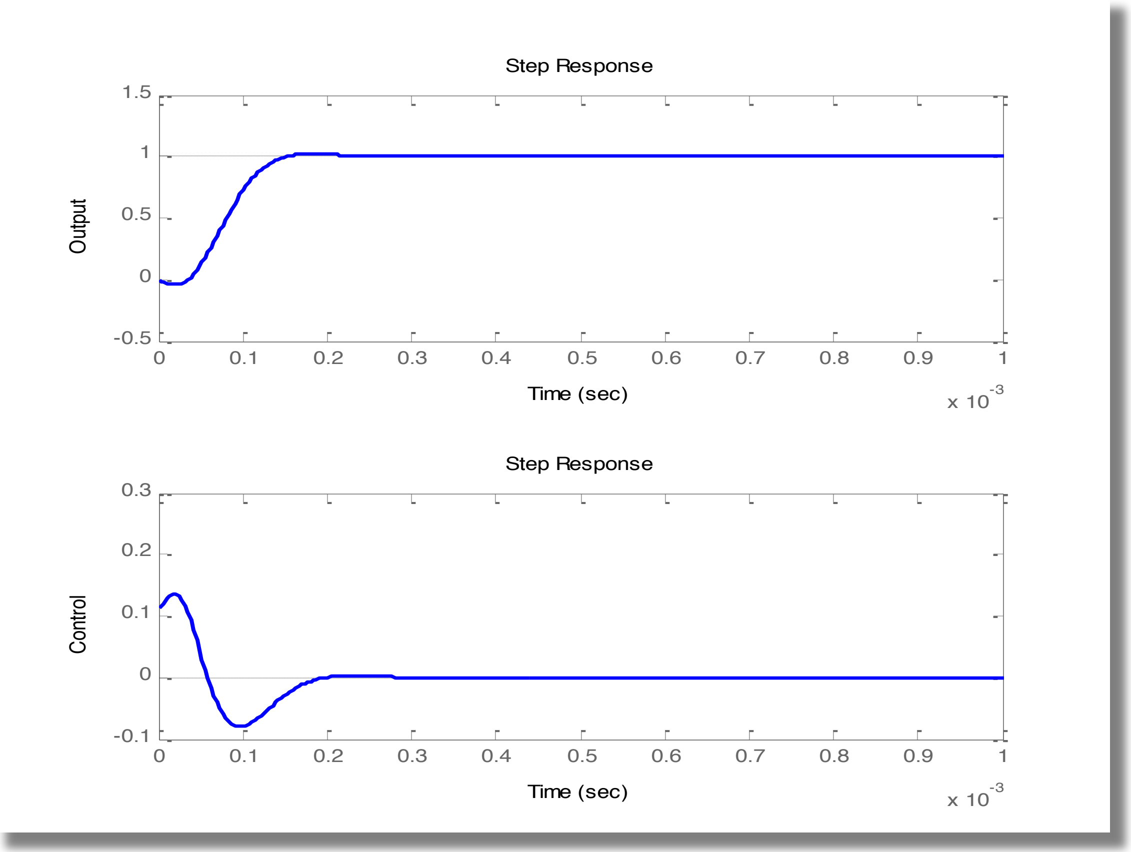
Simple solution: static adjustment of feedforward gain

$$Y(0) = G_c(0) L_r R(0) \Rightarrow L_r = 1/G_c(0)$$



Mikael Johansson mikaelj@ee.kth.se

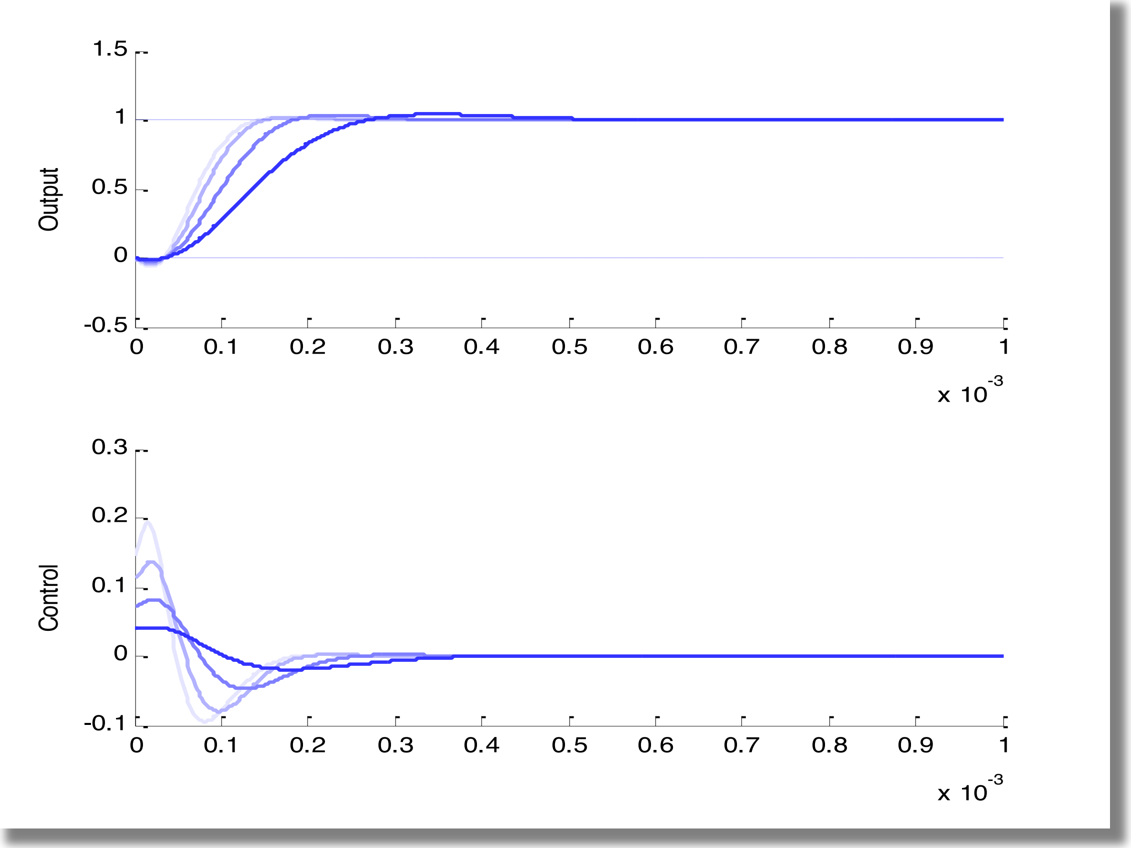
Adjusting feedforward gain



Mikael Johansson mikaelj@ee.kth.se

Shaping the time response

Using $Q_1=I$, $Q_2=\rho$; responses for varying ρ (which is which?)

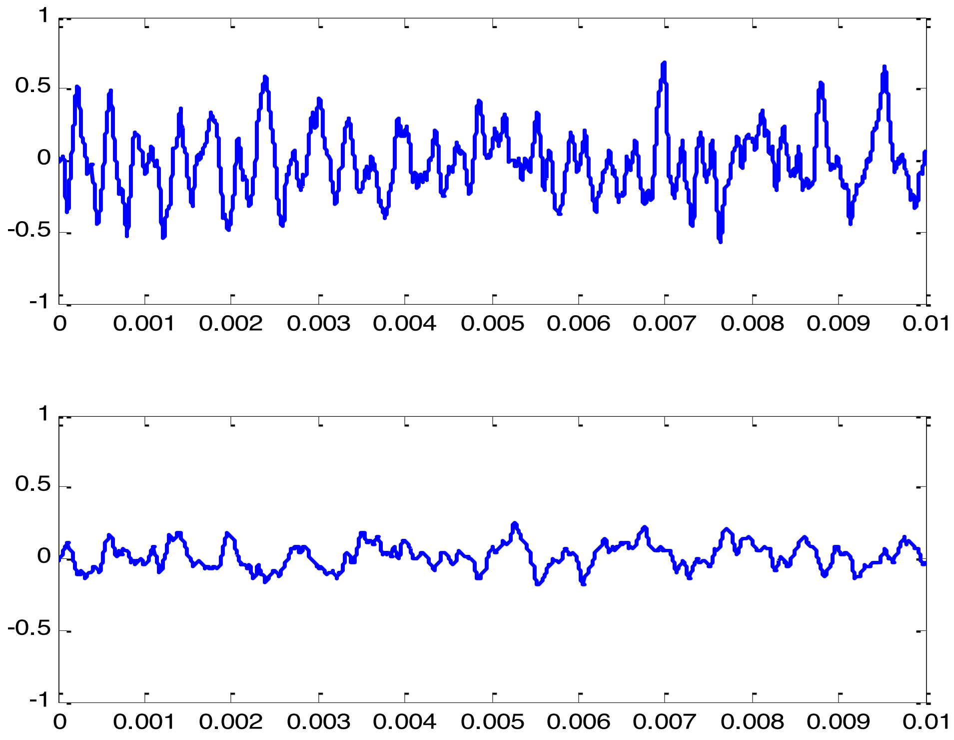


Mikael Johansson mikaelj@ee.kth.se

Suppression of measurement noise

Let R_1=I, R_2=r. Time responses of z to unit variance measurement noise

- Which design corresponds to the larger value of r?

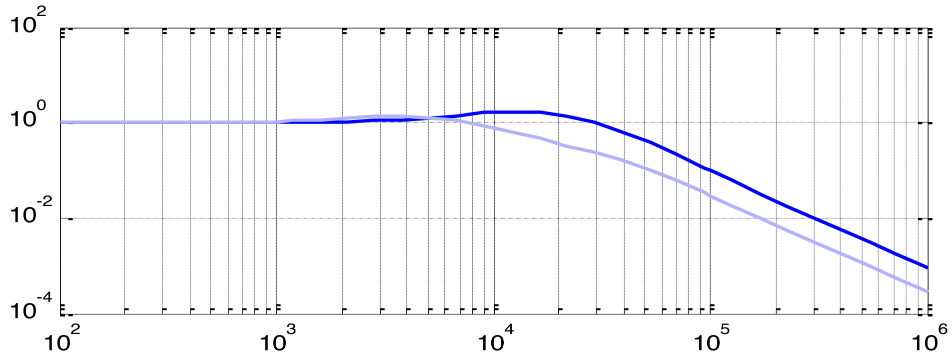


Mikael Johansson mikaelj@ee.kth.se

A loop shaping perspective

Corresponding Bode diagrams of complementary sensitivity (n$\rightarrow$ z)

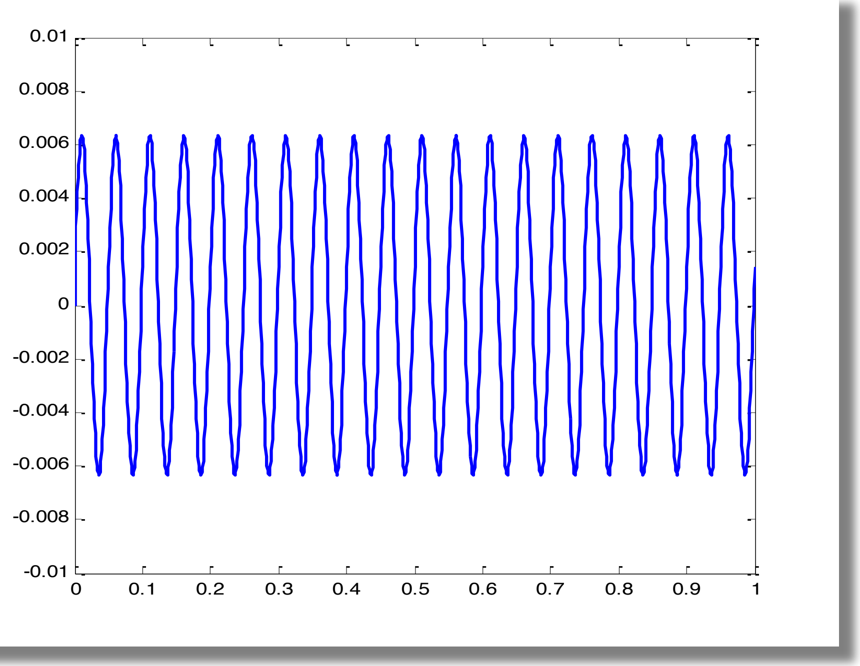
- Which one corresponds to the larger value of r?



Mikael Johansson mikaelj@ee.kth.se

Dealing with output disturbance

Response to sinusoidal output disturbance at 20 Hz

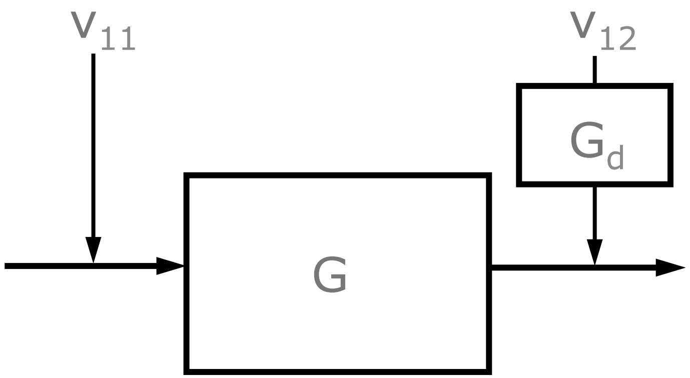


We would like the amplitude to be less than 1E-4.

– How can we achieve this?

Mikael Johansson mikaelj@ee.kth.se

Introducing disturbance model



We will use

$$G_d = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

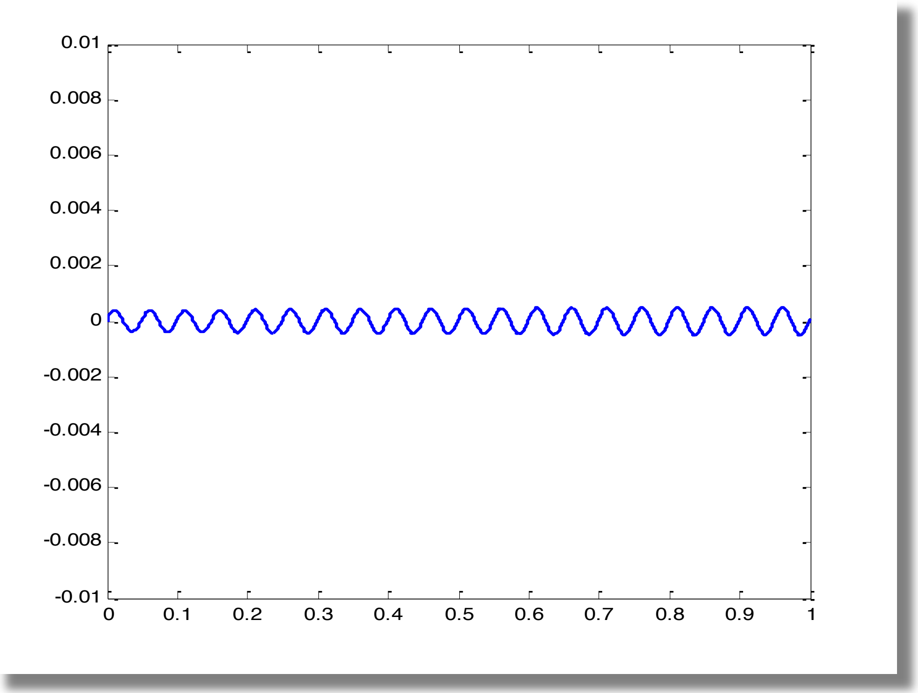
What are the appropriate values for ζ and ω_0?

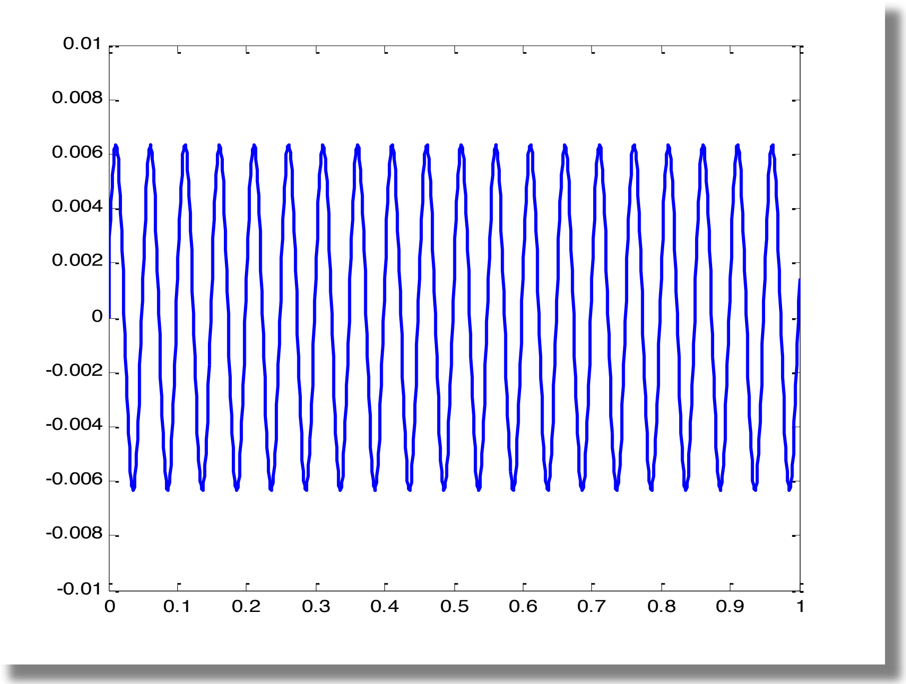
How should we choose R_1?

Mikael Johansson mikaelj@ee.kth.se

Improved disturbance suppression

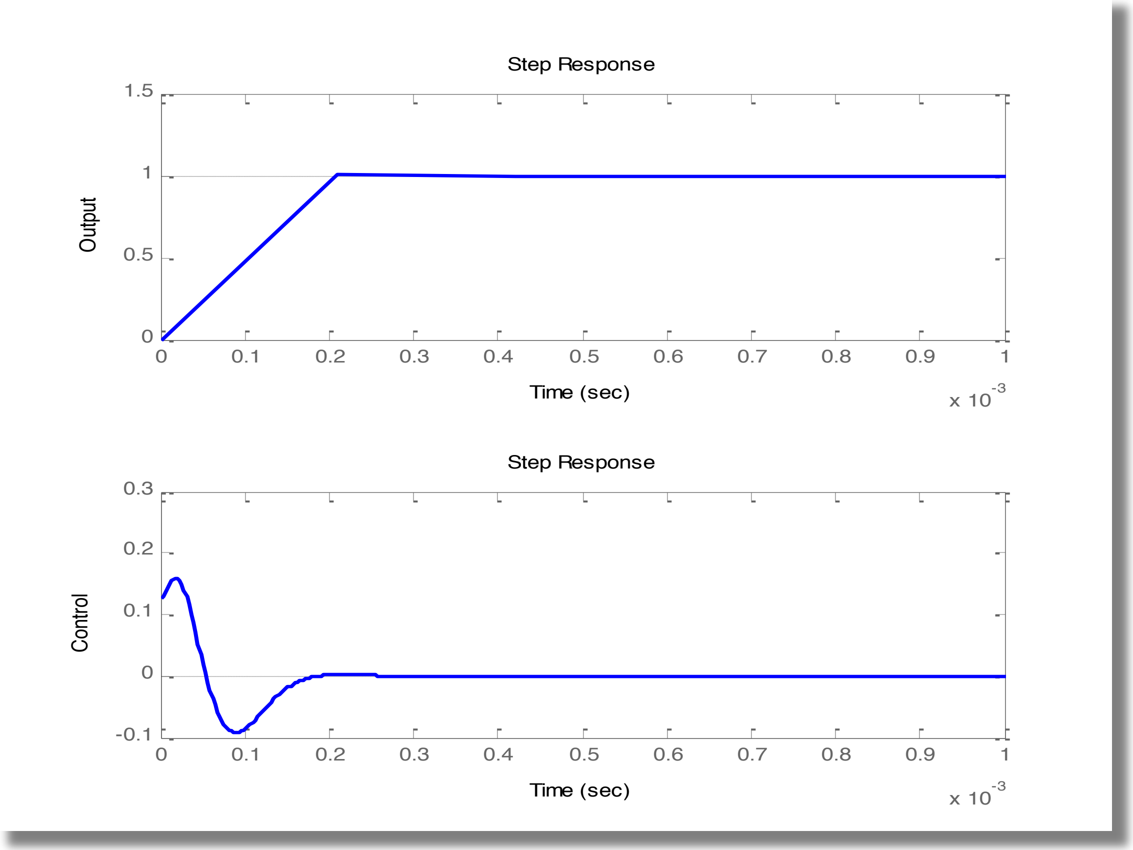
Disturbance response with (left) and without (right) disturbance model





Mikael Johansson mikaelj@ee.kth.se

Reference following of final controller



Which is the order of the final controller (and why?)

Mikael Johansson mikaelj@ee.kth.se

What about an H-infinity design?

Our requirements constrain S and T – a standard problem!

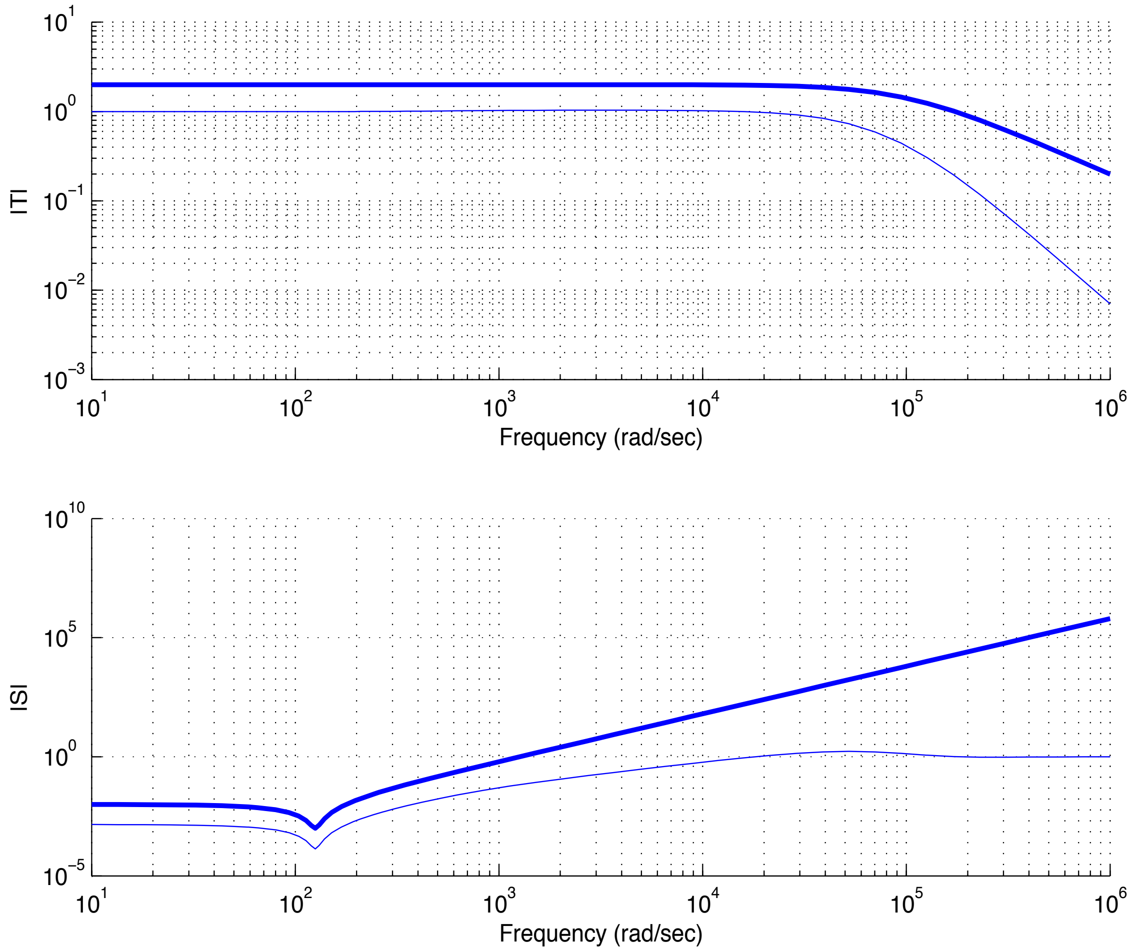
What are reasonable weights?

Which is the corresponding extended system?

What is the order of the resulting controller?

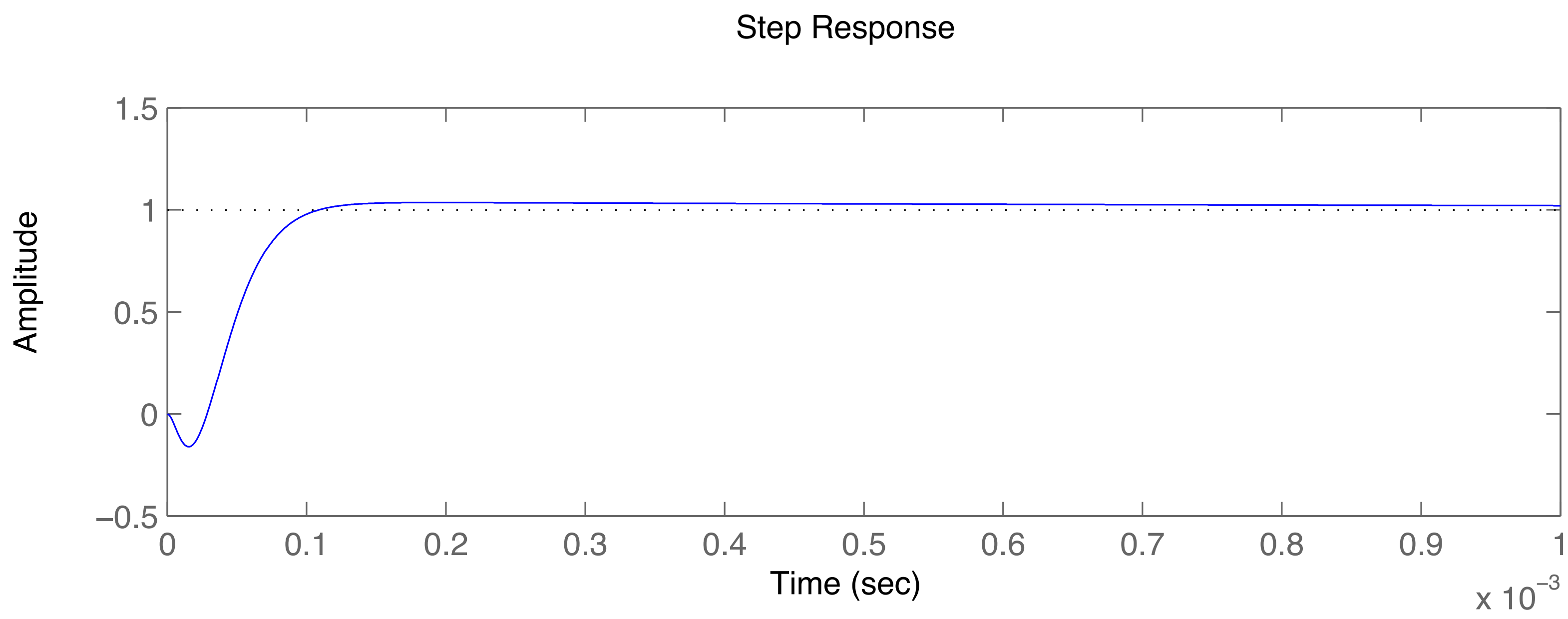
Mikael Johansson mikaelj@ee.kth.se

Result of H-infinity design



Mikael Johansson mikaelj@ee.kth.se

Result of H-infinity design



Raw step-response. As for LQG, can be improved by feed-forward.

Mikael Johansson mikaelj@ee.kth.se

Relation to H_2-optimal control

Definition. The H_2 norm of a system G is defined as

$$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(i\omega)|_2^2 \, d\omega$$

Now, assume that w is white noise with $\Phi_w(\omega)=I$. Then

$$\begin{aligned}\|z\|_2^2 &= \int_{-\infty}^{\infty} y^2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Z(i\omega)|^2 d\omega = \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(i\omega)W(i\omega)|_2^2 \, d\omega = \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\{G(i\omega)W(i\omega)W^T(-i\omega)G^T(-i\omega)\}\, d\omega = \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\{G(i\omega)G^T(-i\omega)\}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(i\omega)|_2^2 \, d\omega\end{aligned}$$

Mikael Johansson mikaelj@ee.kth.se

Relation to H_2-optimal control

Consequence: minimizing the H_2-norm is equivalent to minimizing the influence of a white noise input w on z.

$$\|z\|_2^2 = \|Mx + Du\|_2^2 = \|Mx\|_2^2 + \|u\|_2^2 =$$
$$= \int_{-\infty}^{\infty} x^T M^T M x + u^T u \, dt$$

The LQG problem in standard form with $Q_1=Q_2=R_1=R_2=I$

Summary

- Linear quadratic control review
  - Solution structure, Riccati equations and tuning knobs
- Design example: radial control of DVD player
  - Reference following
  - Trading off state vs control energy
  - Influencing sensitivity to noise
  - Shaping the response to non-white disturbances
- Addressing the same problem using H-infinity
- A relation between LQG and H_2-optimal control

Mikael Johansson mikaelj@ee.kth.se