2E1252
Control Theory and Practice

Lecture 9:
LQG design example and relation to H_2

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Today's lecture

- Linear quadratic control review
- A design example: radial control of DVD servo
- Relation to H_2-optimal control

Linear quadratic Gaussian control

Model: linear system with white noise

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t) + Nv_1(t)$$
$$y(t) = Cx(t) + v_2(t)$$
$$z(t) = Mx(t)$$

where v_1, v_2 are white noise with

$$\text{cov}([v_1,\ v_2]) = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$$

Objective: minimize effect of v on z, punish control cost

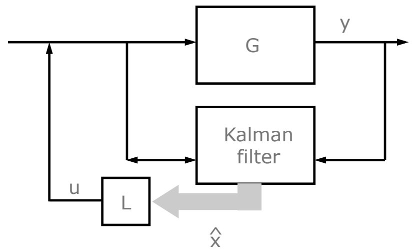
$$J = \mathbf{E}\left\{\lim_{T\to\infty} \int_0^T [z^T Q_1 z + u^T Q_2 u]\ dt\right\}$$

LQG: Linear system, Quadratic cost, Gaussian noise

Solution structure

Optimal solution satisfies *separation principle*, composed of

- Optimal linear state feedback (Linear-quadratic regulator)
- Optimal observer (Kalman filter)



Optimal solution

State feedback

$$u(t) = -L\hat{x}(t) = -Q_2^{-1}B^TS\hat{x}(t)$$

where S is the solution to the algebraic Riccati equation

$$A^TS + SA + M^TQ_1M - SBQ_2^{-1}B^TS = 0$$

Kalman filter

$$\dot{\hat{x}}(t) = Ax(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

where $K=(PC^T+NR_{12})R_2^{-1}$ and P is the solution to

$$AP + PA^T + NR_1N^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T = 0$$

Example: LQR for scalar system

Scalar linear system

$$\dot{x}(t) = ax(t) + u(t), \qquad y(t) = x(t)$$

with cost

$$J = \int_0^\infty [x^2 + \rho u^2]\, dt$$

Riccati equation

$$2as + 1 - s^2/\rho = 0$$

has solutions

$$s = a\rho \pm \sqrt{(\rho a)^2 + \rho}$$

so the optimal feedback law is

$$u = -(s/\rho)x = -(a + \sqrt{a^2 + 1/\rho})x$$

Example: LQR for scalar system

Closed loop system

$$\dot{x}(t) = -(\sqrt{a^2 + 1/\rho})x(t)$$

Note: if system is unstable (a>0), then

- if control is expensive $\rho \to \infty$ then the minimum control input to stabilize the plant is obtained with the input u=-2|a|x, which moves the unstable pole to its mirror image –a
- if control is cheap ($\rho \to 0$), the closed loop bandwidth is roughly $1/\sqrt{\rho}$

Example: scalar system Kalman filter

Scalar linear system

$$\dot{x}(t) = ax(t) + u(t) + v_1(t), \qquad y(t) = x(t) + v_2(t)$$

with covariances $E\{v_1^2\}=R_1$, $E\{v_2^2\}=R_2$, $E\{v_1v_2\}=0$.

Riccati equation

$$2ap + r_1 - p^2/r_2 = 0$$

gives

$$k = a + \sqrt{a^2 + r_1/r_2}$$

and estimation error dynamics

$$\frac{d}{dt}\tilde{x}(t) = -(\sqrt{a^2 + r_1/r_2})\tilde{x}$$

Interpretation: measurements discarded if too noisy.

The tuning knobs

State and control weights:
- Trade-off between control effort and state errors (Q_1=1, Q_2=ρ gives closed-loop bandwidth $\sim 1/\sqrt{\rho}$)
- Rule of thumb: start with diagonal Q_1, i.e.

$$z^T Q_1 z = q_{11} z_1^2 + \cdots + q_{kk} z_k^2$$

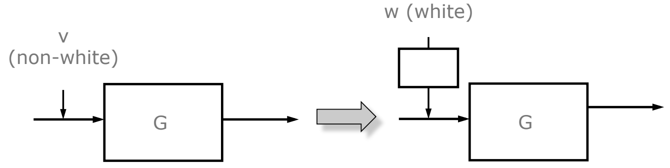
where q_{ll} is inversely proportional to maximum allowed value of z_l (similarly with Q_2)

Noise covariance matrices
- Trade-off between sensitivity to process and measurement noise

(estimator bandwidth $\sim \sqrt{r_1/r_2}$)

White noise inputs

No serious restriction:
- practically all disturbance spectra can be realized as filtered white noise
- need to augment system model with disturbance model



The servo problem

Preferred way: augment system with reference model

$$\frac{d}{dt} x_{\text{ref}}(t) = A_{\text{ref}} x_{\text{ref}}(t) + B r_z(t)$$

$$e(t) = x_{\text{ref}}(t) - x(t)$$

where the reference model states are measurable, and use

$$J = \mathbf{E}\left\{ \lim_{T\to\infty} \int_0^T [e^T Q_1 e + u^T Q_2 u]\, dt \right\}$$

However, when r is assumed to be constant the solution is

$$u(t) = -L\hat{x}(t) + L_{\text{ref}} r_z(t)$$

where L_{ref} is determined so that static gain of closed-loop (r_z→z) equals the identity matrix

LQG and loop shaping

- LQG: simple to trade-off response-time vs. control effort
  - but what about sensitivity and robustness?
- These aspects can be accounted for using the noise models
  - Sensitivity function: transfer matrix w→z
  - Complementary: transfer matrix n→z

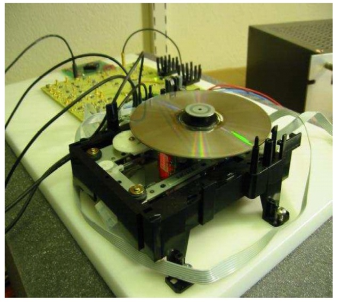
Example: S forced to be small at low frequencies by letting (some component of) w1 affect the input, and let w1 have large energy at low frequencies,

$$w_1(t) = \frac{1}{p+\delta} v_1(t)$$

(delta small, strictly positive, to ensure stabilizability)

Design example

Track following (radial control) in DVD player

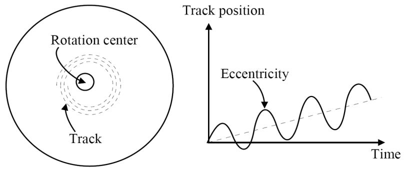


(based on laboratory exercise at LTH)

Design example

Control lens position to follow track

- High bandwidth to allow fast read/write
- Key challenge: eccentricity of tracks on disk
- Sinusoidal disturbance of order 100 track widths!

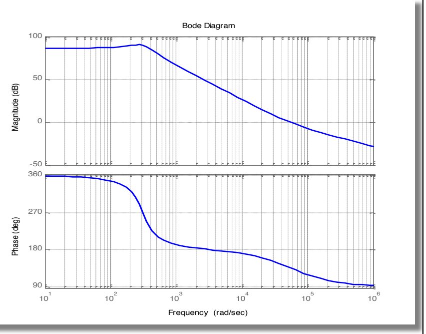


Model

Model identified from real system

$$\dot{x} = \begin{pmatrix} 13.48 & -613.3 \\ 160.4 & -221.7 \end{pmatrix} x + \begin{pmatrix} -9.57 \\ -1046 \end{pmatrix} u$$

$$y = \begin{pmatrix} 3354 & 5.40 \end{pmatrix} x$$

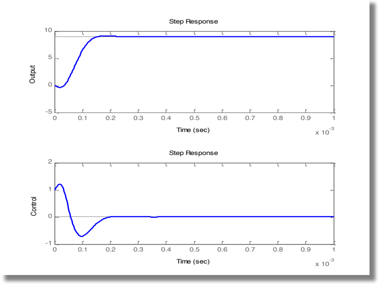


An initial design

Use

$$z = y, \quad Q_1 = 1, \quad Q_2 = 1, \quad R_1 = I, \quad R_2 = 1$$

consider response to unit step input in reference:

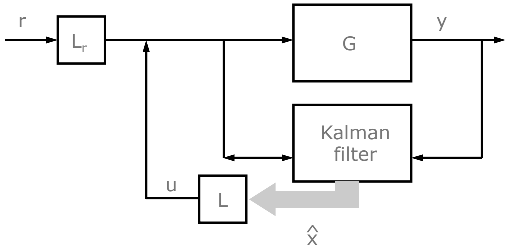


What is wrong?

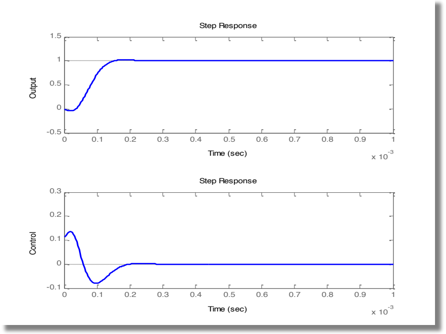
Adjusting feedforward gain

Simple solution: static adjustment of feedforward gain

$$Y(0) = G_c(0) L_r R(0) \Rightarrow L_r = 1/G_c(0)$$

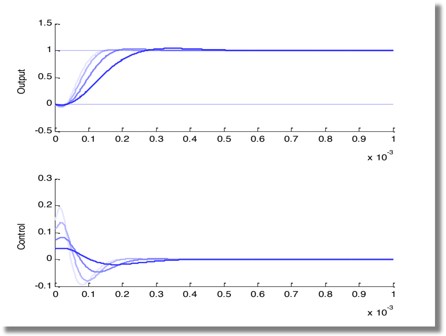


Adjusting feedforward gain



Shaping the time response

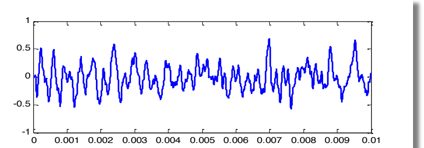
Using $Q_1 = I$, $Q_2 = \rho$; responses for varying ρ (which is which?)

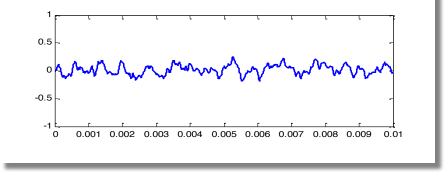


Suppression of measurement noise

Let $R_1 = I$, $R_2 = r$. Time responses of z to unit variance measurement noise

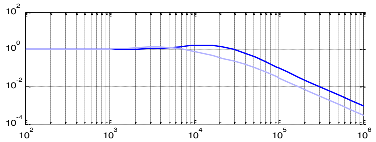
– Which design corresponds to the larger value of r?





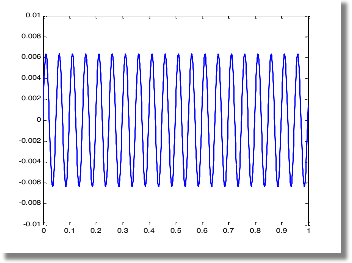
A loop shaping perspective

Corresponding Bode diagrams of complementary sensitivity ($n \rightarrow z$)
- Which one corresponds to the larger value of r?



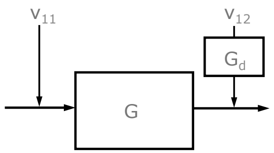
Dealing with output disturbance

Response to sinusoidal output disturbance at 20 Hz



We would like the amplitude to be less than 1E-4.
- How can we achieve this?

Introducing disturbance model



We will use

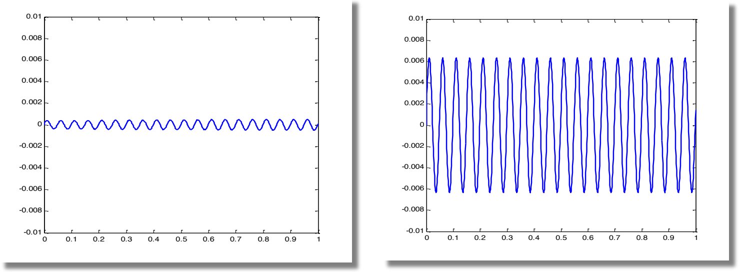
$$G_d = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

What are the appropriate values for ζ and ω_0?

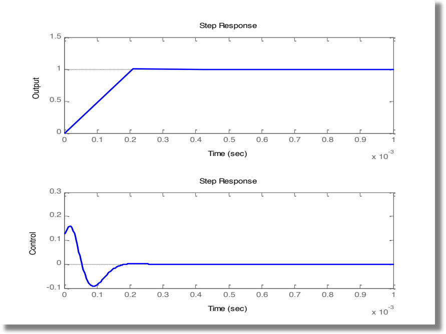
How should we choose R_1?

Improved disturbance suppression

Disturbance response with (left) and without (right) disturbance model



Reference following of final controller



Which is the order of the final controller (and why?)

What about an H-infinity design?

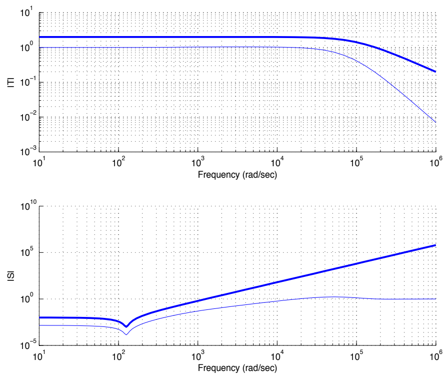
Our requirements constrain S and T – a standard problem!

What are reasonable weights?

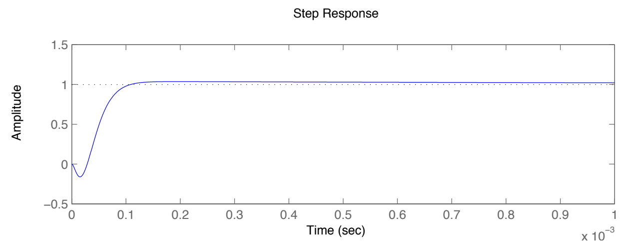
Which is the corresponding extended system?

What is the order of the resulting controller?

Result of H-infinity design



Result of H-infinity design



Raw step-response. As for LQG, can be improved by feed-forward.

Relation to H_2-optimal control

Definition. The H_2 norm of a system G is defined as

$$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(i\omega)|_2^2 \, d\omega$$

Now, assume that w is white noise with $\Phi_w(\omega)=I$. Then

$$\begin{aligned}\|z\|_2^2 &= \int_{-\infty}^{\infty} y^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Z(i\omega)|^2 d\omega = \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(i\omega)W(i\omega)|_2^2 \, d\omega = \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\{G(i\omega)W(i\omega)W^T(-i\omega)G^T(-i\omega)\}\, d\omega = \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\{G(i\omega)G^T(-i\omega)\}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(i\omega)|_2^2 \, d\omega\end{aligned}$$

Relation to H_2-optimal control

Consequence: minimizing the H_2-norm is equivalent to minimizing the influence of a white noise input w on z.

$$\begin{aligned}\|z\|_2^2 &= \|Mx + Du\|_2^2 = \|Mx\|_2^2 + \|u\|_2^2 = \\ &= \int_{-\infty}^{\infty} x^T M^T M x + u^T u \, dt\end{aligned}$$

The LQG problem in standard form with $Q_1=Q_2=R_1=R_2=I$

Summary

- Linear quadratic control review
  - Solution structure, Riccati equations and tuning knobs
- Design example: radial control of DVD player
  - Reference following
  - Trading off state vs control energy
  - Influencing sensitivity to noise
  - Shaping the response to non-white disturbances
- Addressing the same problem using H-infinity
- A relation between LQG and H_2-optimal control