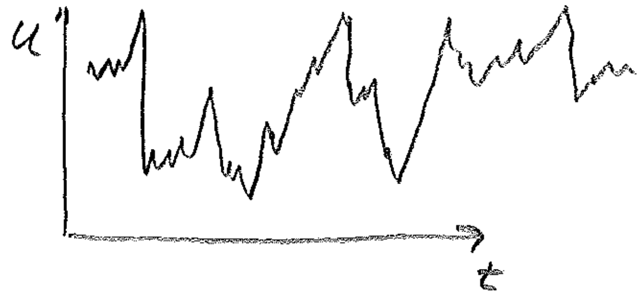
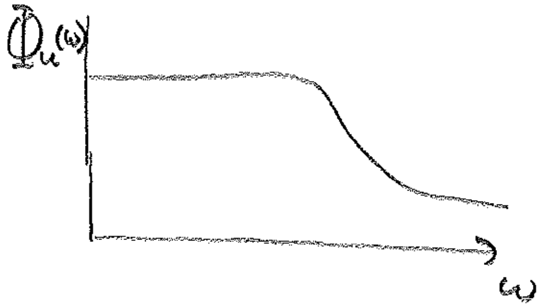
Theory:

Stochastic signals:

Described by a spectrum

Ex:





White noise:

$\Phi_v(\omega) = R$ constant matrix (the intensity)

"energy is equally distributed over all frequencies"

Filtered signals:

If $Y(s) = G(s)U(s)$

$\Phi_u(\omega)$ - spectrum of u

then

$$\Phi_y(\omega) = G(i\omega)\Phi_u(\omega)(G(i\omega))^*$$

Spectral factorization: Signals as filtered white noise

$\Phi_u(\omega)$ - rational in ω^2 $\Rightarrow$

$\Phi_u(\omega) = H(i\omega)\, I\, H(i\omega)^*$ $\Rightarrow$

$U(s) = H(s)\, V(s)$ ← white noise with intensity I.

Covariance of states:

$\dot{x} = Ax + Bv$ ← white noise with intensity R

$y = Cx$

Π_x = Covariance of states x

$A\Pi_x + \Pi_x A^T + BRB^T = 0$

$\Pi_y = C\Pi_x C^T$ = Covariance of output y.

Kalman filter: Optimal observer

if $\dot{x} = Ax + Bu + Nv_1$

$y = Cx + Du + v_2$

$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ - white noise with $R = \begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix}$

Then

$$\dot{\hat{x}} = A\hat{x} + Bu + K[y - Cx - Du]$$

with $K = [PC^T + NR_{12}]R_2^{-1}$ minimize $E\tilde{x}\tilde{x}^T$.

$P = P^T \geq 0$ is the solution to

$$AP + PA^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T + NR_1N^T = 0$$

LQ-controller:

min $\lim_{\tau \to \infty} \frac{1}{\tau}\int_0^\tau (zQ_1z^T + uQ_2u^T)\,dt$

$\dot{x} = Ax + Bu$

$z = Mx$

we want to control z.

then $u = -Lx$ where $L = Q_2^{-1}B^TS$

$S = S^T \geq 0$ is the solution to

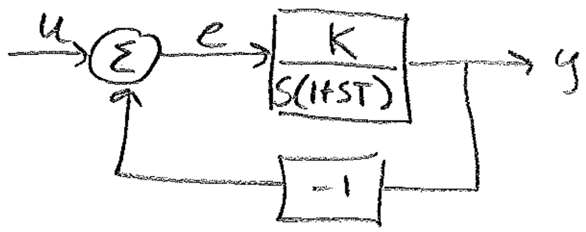
$$A^TS + SA + M^TQ_1M - SBQ_2^{-1}B^TS = 0$$

LQG-controller:

Combine above to

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x} - Du) & \leftarrow \text{kalman filter} \\ u = -L\hat{x} & \leftarrow \text{LQ-controller} \end{cases}$$

9.1 Given the system



where u is a stochastic signal with spectrum

$$\Phi_u(\omega) = \frac{\omega_0^2}{\omega_0^2 + \omega^2}$$

a) What is the spectrum of e?

$E(s) = S(s)U(s)$ with $S(s) = \left(1 + \frac{K}{s(1+sT)}\right)^{-1}$

$$= \frac{s(1+sT)}{s(1+sT)+K}$$

$$\Phi_e(\omega) = S(i\omega)\,\Phi_u(\omega)\,S(-i\omega) =$$

$$= \frac{i\omega - \omega^2 T}{i\omega - \omega^2 T + K} \cdot \frac{\omega_0^2}{\omega_0^2 + \omega^2} \cdot \frac{-i\omega - \omega^2 T}{-i\omega - \omega^2 T + K} =$$

$$= \underline{\underline{\frac{\omega^4 T^2 + \omega^2}{(K - \omega^2 T)^2 + \omega^2} \, \frac{\omega_0^2}{\omega_0^2 + \omega^2}}}$$

b) For which K is the variance $Ee^2(t)$ minimized?

(I) Spectral factorization $\Rightarrow$

$$E(s) = S(s)\, \underbrace{H(s)\, V(s)}_{U(s)}$$ ← white noise

(II) State space form: $S(s)\, H(s)$ $\Rightarrow$

$$\dot{x} = Ax + Bv$$
$$e = Cx$$

(III) Compute state covariance matrix P.

(IV) Variance $Ee^2(t) = CPC^T$

(V) minimize $Ee^2(t)$.

Spectral factorization:

u have spectrum $\Phi_u(\omega) = \dfrac{\omega_0^2}{\omega_0^2 + \omega^2} = H(i\omega) H(-i\omega)$

$$\frac{\omega_0^2}{\omega_0^2 + \omega^2} = \frac{\omega_0}{\omega_0 + i\omega} \cdot \frac{\omega_0}{\omega_0 - i\omega} \quad \text{(conjugate rule)} \Rightarrow$$

$$H(s) = \frac{\omega_0}{\omega_0 + s}$$

$$\Rightarrow E(s) = \frac{s(1+sT)}{s(1+sT)+K} \cdot \frac{\omega_0}{\omega_0 + s}\, V(s)$$

State space:

We try with the observable canonical form

$\left(\text{since} \quad C = (1\ 0\ 0) \Rightarrow E e^2(t) = (1\ 0\ 0)\, P \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = P_{11} \right)$

$$E(s) = \frac{\omega_0 s^2 + \frac{\omega_0}{T} s}{s^3 + (\omega_0 + 1/T) s^2 + \frac{\omega_0 + K}{T} s + \frac{\omega_0 K}{T}} V(s)$$

$$\Rightarrow \quad \dot{x} = \begin{pmatrix} -(\omega_0 + 1/T) & 1 & 0 \\ -\frac{\omega_0 + K}{T} & 0 & 1 \\ -\frac{\omega_0 K}{T} & 0 & 0 \end{pmatrix} x + \begin{pmatrix} \omega_0 \\ \frac{\omega_0}{T} \\ 0 \end{pmatrix} V$$

$$e = (1\ 0\ 0) x$$

Covariance of states:

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{pmatrix}$$

Such that

$$AP + PA^T + BB^T = 0$$

Doing this multiplication leads to a system of equations

$$\begin{cases} -2(\omega_0 + 1/T) P_{11} + 2P_{12} + \omega_0^2 = 0 \\ -(\omega_0 + 1/T) P_{13} + P_{23} - \frac{\omega_0 K}{T} P_{11} = 0 \\ -\frac{2(\omega_0 + K)}{T} P_{12} + 2P_{23} + \frac{\omega_0^2}{T^2} = 0 \\ -2\frac{\omega_0 K}{T} P_{13} = 0 \end{cases}$$

plus five more equations (which we don't need).

$$P_{13} = 0 \Rightarrow P_{23} = \frac{-\omega_0 K}{T} P_{11} \Rightarrow$$

$$P_{12} = \frac{\omega_0 K P_{11} + \frac{\omega_0^2}{2T}}{\omega_0 + K} \Rightarrow$$

$$\underline{P_{11} = \frac{\omega_0^2 (KT + T\omega_0 + 1)}{2(K + T\omega_0^2 + \omega_0)}}$$

Variance of e:

Since we used the observable canonical form we get

$$Ee^2(t) = C^T P C = P_{11}.$$

Minimizing the variance: (For stability we need $K > 0$)

$$\frac{dP_{11}}{dK} = \frac{\omega_0^2 (T^2\omega_0^2 - 1)}{2(K + T\omega_0^2 + \omega)^2}$$

$T > \frac{1}{\omega_0} \Rightarrow \frac{dP_{11}}{dK} > 0 \ \forall K > 0 \quad K \to 0$ best choice

$T < \frac{1}{\omega_0} \Rightarrow \frac{dP_{11}}{dK} < 0 \ \forall K > 0 \quad K \to \infty$ best choice

$T = \frac{1}{\omega_0} \Rightarrow$ all K are minimizers.

9,3 Create a controller for the system

$\ddot{y}(t) = u(t)$

Such that

$$\int_0^\infty \left(y^2(t) + \eta \cdot u^2(t)\right) dt \quad ; \ \eta > 0$$

is minimized. Assume y & $\dot{y}$ are measured. What is the closed loop poles?

What happens with u as η is decreased.

We note that we can write

$$\int_0^\infty (y^2 + \eta u^2) dt = \int_0^\infty (y Q_1 y + u Q_2 u) dt \qquad Q_1 = 1 \quad Q_2 = \eta$$

$\Rightarrow$ We can use LQ methodology to find the controller.

State space:

Let $x = \begin{pmatrix} y \\ \dot{y} \end{pmatrix} \Rightarrow$

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = (1 \ 0) x$$

We want to control the thing we measure $\Rightarrow z = y \Rightarrow M = C$ and we have no noise.

The LQ-controller is $u = -Lx$ (state feedback)

With $L = Q_2^{-1} B^T S = \frac{1}{\eta}\begin{pmatrix}0 & 1\end{pmatrix} S$

$S = S^T \geq 0$ is the solution to

$$A^T S + SA + M^T Q_1 M - S B Q_2^{-1} B^T S = 0$$

We have

$$\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}S_{11} & S_{12}\\ S_{12} & S_{22}\end{pmatrix} + \begin{pmatrix}S_{11} & S_{12}\\ S_{12} & S_{22}\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + \overbrace{\begin{pmatrix}1\\ 0\end{pmatrix}\cdot 1\cdot\begin{pmatrix}1 & 0\end{pmatrix}}^{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}} -$$

$$\underbrace{\begin{pmatrix}S_{11} & S_{12}\\ S_{12} & S_{22}\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}}_{\begin{pmatrix}S_{12}\\ S_{22}\end{pmatrix}}\frac{1}{\eta}\underbrace{\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}S_{11} & S_{12}\\ S_{12} & S_{22}\end{pmatrix}}_{\begin{pmatrix}S_{12} & S_{22}\end{pmatrix}} = 0$$

$$= \begin{pmatrix}0 & 0\\ S_{11} & S_{12}\end{pmatrix} + \begin{pmatrix}0 & S_{11}\\ 0 & S_{12}\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} - \frac{1}{\eta}\begin{pmatrix}S_{12}^2 & S_{12}S_{22}\\ S_{12}S_{22} & S_{22}^2\end{pmatrix} = 0$$

$\Rightarrow$ System of equations

$$\begin{cases} 0 + 0 + 1 - \dfrac{S_{12}^2}{\eta} = 0\\ 0 + S_{11} + 0 - \dfrac{S_{12}S_{22}}{\eta} = 0\\ S_{11} + 0 + 0 - \dfrac{S_{12}S_{22}}{\eta} = 0\\ S_{12} + S_{12} + 0 - \dfrac{S_{22}^2}{\eta} = 0\end{cases}$$

Solving this yields:

$$s_{12} = \pm \eta^{1/2}$$

$$s_{22} = \pm \sqrt{2}\, \eta^{3/4} \qquad \Rightarrow \quad S = \underset{(-)}{\pm} \begin{pmatrix} \sqrt{2}\,\eta^{1/4} & \eta^{1/2} \\ \eta^{1/2} & \sqrt{2}\,\eta^{3/4} \end{pmatrix}$$

$$s_{11} = \pm \sqrt{2}\, \eta^{1/4}$$

it should be positive semidefinite.

Hence $L = \frac{1}{\eta} \cdot (0 \;\; 1) \begin{pmatrix} \sqrt{2}\,\eta^{1/4} & \eta^{1/2} \\ \eta^{1/2} & \sqrt{2}\,\eta^{3/4} \end{pmatrix} = (\eta^{-1/2} \;\; \sqrt{2}\,\eta^{-1/4})$

Let $\alpha = \eta^{-1/4} \Rightarrow L = (\alpha^2 \;\; \sqrt{2}\,\alpha)$

Closed loop poles: are eigenvalues of $(A - BL)$

$$\det\left(sI - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \alpha^2 & \sqrt{2}\,\alpha \end{pmatrix}\right) = 0 \Rightarrow$$

$$\det\begin{pmatrix} s & -1 \\ \alpha^2 & s + \sqrt{2}\,\alpha \end{pmatrix} = 0 \Rightarrow$$

$$s^2 + \sqrt{2}\,\alpha s + \alpha^2 = 0 \Rightarrow$$

$$s = -\frac{\alpha \pm i\alpha}{\sqrt{2}} = -\frac{1 \pm i}{\sqrt{2}\,\eta^{1/4}}$$

If η is decreased, L gets larger $\Rightarrow u$ larger

We also see that the poles move further away from the origin $\Rightarrow$ faster system.

9,7 Consider the system

$$Z(s) = G(s)(U(s) + V(s)) \qquad G(s) = \frac{1}{s+1}$$
$$Y(s) = Z(s) + E(s)$$

Where V & e are disturbances with spectra $\Phi_v(\omega) = r_1$ $\quad \Phi_e(\omega) = 1$

We minimize $\int (q_1 z^2 + u^2)dt$

a) Find the loop transfer function $G F_y$

Using e.g. the observable canonical form we get a state space description

$$\begin{cases} \dot{x} = -x + u + v \\ z = x \\ y = x + e \end{cases} \Rightarrow \quad \begin{matrix} A = -1 & B = 1 & N = 1 \\ M = 1 & C = 1 & \end{matrix}$$

The LQG-controller becomes

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}) & \leftarrow \text{kalman filter} \\ u = -L\hat{x} & \leftarrow \text{State feedback} \end{cases}$$

<u>The Kalman filter:</u>

We assume that V & e are uncorrelated, then $\begin{bmatrix} V \\ e \end{bmatrix}$ is white noise with intensity

$$R = \begin{pmatrix} r_1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_1 & R_{12} \\ R_{12} & R_2 \end{pmatrix}$$

then we get

$K = (PC^T + NR_{12})R_2^{-1}$

where $P = P^T \geq 0$ is the solution to

$AP + PA^T + NR_1N^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T = 0$

$-P - P + 1 \cdot r_1 1 - (P + 0)\, 1^{-1} (P + 0)^T = 0 \Rightarrow$

$$P^2 + 2P - r_1 = 0 \Rightarrow P = -1 (\pm) \sqrt{1 + r_1}$$

$\Rightarrow K = ((-1 + \sqrt{1 + r_1}) \cdot 1 + 0) \cdot 1^{-1} = -1 + \sqrt{1 + r_1}$

<u>The state feedback:</u>

We want to minimize

$$\int (q_1 z^2 + u^2)\,dt \Rightarrow Q_1 = q_1 \quad Q_2 = 1$$

The state feedback is then

$L = Q_2^{-1} B^T S$

where $S = S^T \geq 0$ is the solution to

$A^T S + SA + M^T Q_1 M - SBQ_2^{-1} B^T S = 0$

$-S - S + 1 \cdot q_1 \cdot 1 - S \cdot 1 \cdot 1^{-1} \cdot 1 \cdot S = 0$

$$S^2 + 2S - q_1 = 0 \Rightarrow S = -1 (\pm) \sqrt{1 + q_1}$$

$\Rightarrow L = 1^{-1} \cdot 1 \cdot (-1 + \sqrt{1 + q_1}) = -1 + \sqrt{1 + q_1}$

We want the loop transfer function GF_y so we need $F_y(s)$

$F_y(s)$: Substitute $\hat{x} = \frac{-u}{L}$ in the kalman filter

$$\dot{\hat{x}} = -\hat{x} + u + K(y - \hat{x})$$ to get

$$\dot{u} = -u(1+L+K) - LKy$$

taking the laplace inverse transform $\Rightarrow$

$$sU(s) = -U(s)(1+K+L) - LK\,Y(s) \Rightarrow$$

$$U(s) = \underbrace{\frac{-LK}{s+1+K+L}}_{-F_y} Y(s)$$

Thus, the loop transferfunction becomes

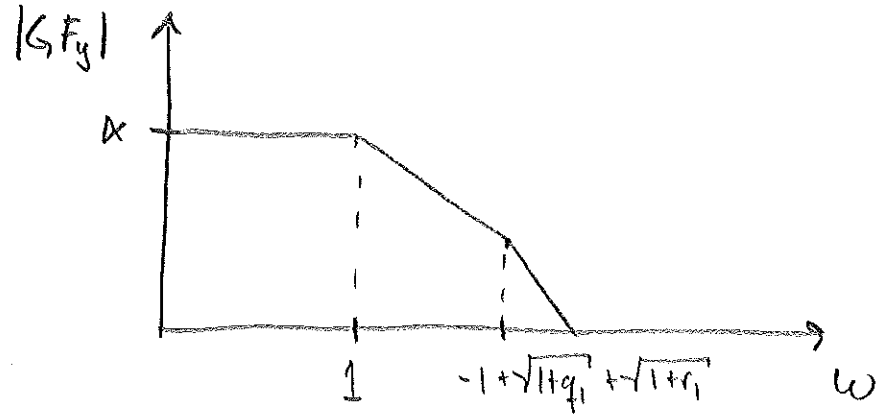
$$G(s)F_y(s) = \frac{1}{s+1} \cdot \frac{(-1+\sqrt{1+q_1})(-1+\sqrt{1+r_1})}{s-1+\sqrt{1+q_1}+\sqrt{1+r_1}}$$

b) What is the difference between r_1 and q_1 in $G(s)F_y(s)$?

None, they affect the loop gain in the same way.

Scetch $G F_y$. What happens if r_1 or $q_1 \to \infty$

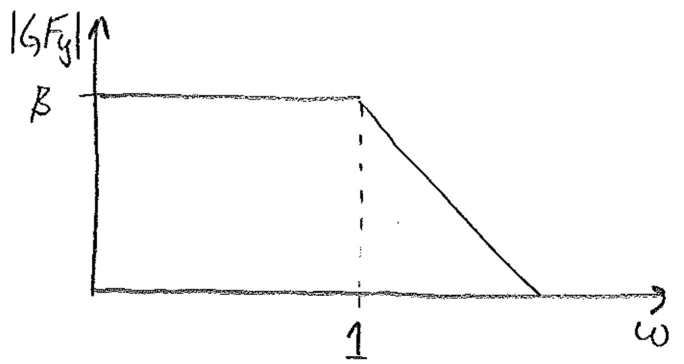
c) We get two breakpoints from the poles at $\omega = 1$ & $\omega = -1+\sqrt{1+q_1}+\sqrt{1+r_1}$



$$\alpha = \frac{(-1+\sqrt{1+r_1})(-1+\sqrt{1+q_1})}{-1+\sqrt{1+q_1}+\sqrt{1+r_1}}$$

We check $r_1 \to \infty$

$$\lim_{r_1\to\infty} G F_y = \frac{-1+\sqrt{1+q_1}}{s+1} \cdot \underbrace{\lim_{r_1\to\infty} \frac{-1+\sqrt{1+r_1}}{s-1+\sqrt{1+q_1}+\sqrt{1+r_1}}}_{=1} = \frac{-1+\sqrt{1+q_1}}{s+1}$$



$$\beta = -1+\sqrt{1+q_1}$$

We can then use the remaining variable q_1 to shape the loop (move β up or down).